# LECTURE NOTES ON K-THEORY

## 1. Lecture 1

1.1. **Paracompact Spaces.** Let $X$ be a topological space. A collection of subsets $(A_i)_{i\in I}$ is said to be **locally finite** if for every $x \in X$ there is an open neighborhood $U$ such that $U$ intersects at most finitely many of the $A_i$s nontrivially. A topological space is **paracompact** if every open cover admits a locally finite subcover. Clearly compact spaces are paracompact.

We observe that if $(A_i)$ is locally finite then $\bigcup_i \overline{A_i} = \overline{\bigcup_i A_i}$. The following are standard exercises in point-set topology.

**Proposition 1.1.** *Let $X$ be a topological space.*

(1) *If $X$ is paracompact and hausdorff ($T_2$), then $X$ is normal ($T_4$).*
(2) *If $X$ is regular ($T_3$) and second countable, then $X$ is paracompact.*
(3) *If $X$ is locally compact, second countable, and hausdorff, then $X$ is paracompact and* $\sigma$**-compact** *($X$ is a countable union of compact sets).*

A **partition of unity** for $X$ is a collection of continuous maps $\{p_i : X \to [0,1]\}_{i\in I}$ such that the collection of support sets $\mathrm{supp}(p_i) := \overline{p_i^{-1}(0,1]}$ is locally finite and $\sum_i p_i(x) = 1$ for all $x \in X$. We will say that a partition of unity is **subordinate** to a cover of $X$ if each support set is contained in some element of the cover.

The main reason for requiring paracompactness is captured by the following standard fact.

**Proposition 1.2.** *A hausdorff topological space is paracompact if and only if every open cover admits a subordinate partition of unity.*

It turns out that every metrizable space is paracompact (a result of A.H. Stone). However this requires stronger axioms for set theory than ZF with Determined Choice.

1.2. **(Smooth) Manifolds.** Our primary examples of (hausdorff) paracompact spaces will be manifolds. An $n$**-manifold** is a topological space which is second countable and locally homeomorphic to $\mathbb{R}^n$.

1.3. **Basic Examples.** The most basic examples will be the $n$**-spheres**

$$S^n := \{(x_0, x_1, \dots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + x_1^2 + \cdots + x_n^2 = 1\}.$$

**Example 1.3.** The $n$-dimensional **real projective space** $\mathbb{R}P^n$ is defined to be the set of lines through the origin in $\mathbb{R}^{n+1}$.

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Since any line through the origin is described by either of the antipodal points where it intersect the $n$-sphere, a more familiar way the idenitfy $\mathbb{R}P^n$ is as the quotient of $S^n$ by the $\mathbb{Z}/2\mathbb{Z}$-action which sends $(x_0, x_1, \dots, x_n)$ to $(-x_0, -x_1, \dots, -x_n)$. Note that $\mathbb{R}P^1 \cong S^1$. (Pinch $S^1$ to obtain a bouquet of 2 circles, then twist one a half turn and lay it over the other.) However $\mathbb{R}P^n$ and $S^n$ are distinct for $n \geq 2$ since $\pi_1(S^n) \cong \{1\}$ for $n \geq 2$ while $\pi_1(\mathbb{R}P^n) \cong \mathbb{Z}/2\mathbb{Z}$.

There is another natural realization of $\mathbb{R}P^n$ in $M_{n+1}(\mathbb{R})$ as the subset of all matrices $P$ such that $P = P^t = P^2$ and $\operatorname{tr}(P) = 1$ via the map $(x_0, x_1, \dots, x_n) \mapsto [x_i x_j]$. The fact that this is onto follows from the spectral theorem for self-adjoint real-valued matrices.

**Example 1.4.** In general, if $M$ is a manifold (smooth manifold), and $G$ is a group of homeomorphisms (diffeomorphisms) which acts freely and properly discontinuously, then the quotient $M/G$ is again a manifold (smooth manifold). If we drop the asssumption that the group is acting freely, the the resulting quotient is known as an **orbifold**.

Note that we may view $S^{2n-1}$ as the complex $n$-sphere

$$\{(z_0, z_1, \dots, z_n) \in \mathbb{C}^n : |z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = 1\}.$$

This realization leads to a wealth of quotient structures on the odd spheres $S^{2n-1}$ coming from the action on each coordinate by the complex units. This is in contrast to the even spheres where the only nontrivial group which can act freely by homeomorphisms is $\mathbb{Z}/2\mathbb{Z}$.

**Example 1.5.** The $n$-dimensional **complex projective space** $\mathbb{C}P^n$ is the space of all complex lines throught the origin in $\mathbb{C}^{n+1}$.

Similarly to real projective space, $\mathbb{C}P^n$ is realized as the quotient of the complex sphere under the relation $(z_0, z_1, \dots, z_n) \sim (\lambda z_0, \lambda z_1, \dots, \lambda z_n)$, where $\lambda$ is a complex unit. For $n = 1$, there is a bijection from the equivalence classes $[z_0, z_1]$, $z_1 \neq 0$ and $\mathbb{C}$ via $[z_0, z_1] \mapsto z_0/z_1$. In this way, $\mathbb{C}P^1$ is the one point compactification of $\mathbb{C}$ (the Riemann sphere) and the map previously described is the stereographic projection from the point at infinity.

**Example 1.6.** Let $p, q \in \mathbb{N}$. There is a free action of $\mathbb{Z}/p\mathbb{Z}$ on $S^3$, identified as the complex 2-sphere, given by $[1](z_0, z_1) := (e^{2\pi i/p} z_0, e^{2\pi i q/p} z_1)$. The quotient of $S^3$ by his action is known as the **lens space** L(p,q).

The next examples we will cover are Lie groups, specifically the orthogonal and unitary groups, which will play a central role later.

### 1.4. **Grassmannians.**

## 2. Lecture 2

2.1. **Lie Groups.** A **Lie group** $G$ is a group which is also a smooth manifold such that the map $G \times G \ni (x, y) \mapsto x^{-1} y$ is smooth.

Given $v \in TG_e$, one constructs an associated smooth vector field $X_v$ on $G$ by $X_v(g) := (L_g)_* v$. The map $[X_v, X_w](e)$ then defines a nonassociative algebra structure on $TG_e$ which is known as the **Lie algebra** associated to $G$, and is often denoted by $\mathfrak{g}$. The

Lie algebra and the Lie group are essentially two sides of the same coin. For instance, there is canonical map $\exp : \mathfrak{g} \to G$ which is surjective in the case that $G$ is connected and compact.

We turn our attention to some concrete constructions of Lie groups and their Lie algebras.

For each $n$ consided the group $U(n)$ of unitary matrices in $M_n(\mathbb{C})$, i.e., all $A \in M_n(\mathbb{C})$ such that $A^*A = I$ where $*$ is the conjugate transpose. In the real case we have the group $O(n)$ of orthognal matrices in $M_n(\mathbb{R})$, i.e., $A^tA = I$. Note that $O(n)$ embeds as a closed subgroup of $U(n)$, and the $U(n)$ is realizable in $M_{2n}(\mathbb{R})$ (exercise: write this down explicitly). Being a unitary or orthogonal matrix is determined by a family of polynomials in the matrix entries, so it is easy to check that both $O(n)$ and $U(n)$ are (compact) Lie groups.

**Proposition 2.1.** *$U(n)$ is smoothly path connected.*

By the spectral theorem any $A \in U(n)$ is of the form $\exp(iT)$ for $T \in M_n(\mathbb{C})$ self adjoint ($T^* = T$), in other words $A = \exp(T')$ where $(T')^* = -T'$. It is easy to see that the set of these "skew-adjoint" operators is an nonassociative algebra under the commutator $[S,T] = ST - TS$. This is, in fact, the Lie algebra $\mathfrak{u}(n)$ of $U(n)$ and the map $\exp : \mathfrak{u}(n) \to U(n)$ is just normal exponentiation of matrices.

This shows that for all $A = \exp(iT)$, there is a smooth path to the indentity given by the one-parameter subgroup $A_t := \exp(itT)$ for $t \in \mathbb{R}$. It follows that $U(n)$ is smoothly path connected (exercise).

(Un)fortunately, this is not true for $O(n)$, since if $A \in O(n)$, $\det(A) \in \{-1, 1\}$, so there can be no continuous path in $O(n)$ connecting to elements with different determinants. However, this is the only obstruction. Let $SO(n)$ be the **special othogonal group**, the closed subgroup of all orthogonal transformations with determinant 1.

**Proposition 2.2.** *$SO(n)$ is smoothly path connected.*

Most of this is left as an exercise. To begin, embed $O(n)$ in $U(n)$ as the subgroup of all unitaries preserving the canonical real structure on $M_n(\mathbb{C})$. Let $A = \exp(T')$ for some $T' \in M_n(\mathbb{C})$ Since $\det(A) = \exp(\mathrm{tr}(T'))$, we can assume $\mathrm{tr}(T') = 0$, and a little algebra shows that $T'$ realized in the natural way as a matrix in $M_{2n}(\mathbb{R})$ is of the form $T \oplus T$ for some $T \in M_n(\mathbb{R})$ with $T^t = -T$. Thus $A = \exp(T)$ in $M_n(\mathbb{R})$. The rest follows in the same way as smooth path connectivity for $U(n)$. Note that this shows that the set of all matrices of determinant $-1$ is also smoothly path connected.

Note that this shows the Lie algebra $\mathfrak{so}(n)$ is just the skew-symmetric real $n \times n$ matrices where the product is the commutator.

2.2. **Homeomorphisms of Spheres.** Let $S^n$ be the $n$-sphere. The **antipode map** is the map $\alpha : (x_0, x_1, \ldots, x_n) \to (-x_0, -x_1, \ldots, -x_n)$, and the canonical **reflection** is the map $\rho : (x_0, x_1, \ldots, x_n) = (-x_0, x_1, \ldots, x_n)$.

**Proposition 2.3.** *If $n$ is odd then $\alpha$ is homotopic to* id*. If $n$ is even, then $\alpha$ is homotopic to $\rho$.*

To see this, note that $O(n)$ is the also the set of all $S^n$ preserving linear isomorphisms. Clearly, both $\alpha$ and $\rho$ are restrictions of elements in $O(n)$ to the sphere, so we will treat them unambiguously as such. If $n$ is odd, then $\det(\alpha) = 1$, which means there is a path in $SO(n)$ connecting $\alpha$ to the identity. This furnishes the required homotopy in the natural way. In the case that $n$ is even, $\det(\alpha) = -1 = \det(\rho)$ and the same reasoning applies.

**Proposition 2.4.** *Let $\rho : S^n \to S^n$ be the reflection . Then $\rho$ is not homotopic to the identity.*

With a little singular homology theory, the proof is pretty short. It is not so hard to check the that induced homomorphism $\rho_* : H_n(S^n;\mathbb{Z}) \to H_n(S^n;\mathbb{Z}) \cong \mathbb{Z}$ on the n-th homology groups is the map which sends $z$ to $-z$. See (Hatcher, Algebraic Topology, p. 134). Other the other hand, as you probably guessed, $\mathrm{id}_*$ is the identity homomorphism. Homotopic maps induce identical homomorphisms on homology groups.

## 3. Lecture 3

3.1. **Defining Vector Bundles.** Let $B$ be a topological space. A **real $n$-vector bundle** over $B$ is a pair $(E, \xi)$ consisting of a topological space $E$ and a continuous map $\xi : E \to B$ satisfying:

- for each $b \in B$, $\xi^{-1}(b)$ has the structure of a finite-dimensional real vector space.
- for each $b \in B$ there is a neighborhood $U \subset B$ such that $\xi^{-1}(U)$ is homeomorphic to $U \times \mathbb{R}^n$ in a way which respects the linear structure on the fibers.

The space $B$ is referred to as the **base space**, $E$ is the **total space** of the bundle, and $\xi^{-1}(b)$ is the **fiber** over $b$. The second condition on $(E, \xi)$ is referred to as **local triviality**, and is crucial for at least two reasons. First, the idea of a vector bundle should capture the notion of a "continuously varying field of vector spaces" over $B$, so we want to rule out things such as $E = \widehat{B} \times \mathbb{R}^n$ being a vector bundle over $B$ where $\widehat{B}$ is $B$ given the discrete topology. Second, we would like to use vector bundles to build global invariants of the topological space $B$, so we should exclude the possibility of any interesting local structure. We will say that $(E, \xi)$ is an **$n$-bundle** if each fiber has dimension $n$.

Analogously, one can talk about complex vector bundles over $B$. We now describe the category $\mathrm{Vect}_k(B)$ of k-vector bundles over $B$ where $k$ is either $\mathbb{R}$ or $\mathbb{C}$. The objects are in place, so let's turn to the morphisms.

A **morphism** of bundles $f : (E, \xi) \to (E', \xi')$ is a continuous map such that the restrictions $f_b : \xi^{-1}(b) \to (\xi')^{-1}(b)$ are linear for all $b \in B$. The map $f$ is an **isomorphim** if an inverse exists and is also a morphism. A subspace $F \subset E$ is a **subbundle** of $(E, \xi)$ if the restriction $(F, \xi|_F)$ is a vector bundle.

The following observation is extremely useful.

**Proposition 3.1.** *Let $f : (E, \xi) \to (E', \xi')$ be a morphism of vector bundles. Then $f$ is an isomorphism if $f_b$ is an isomorphism for each fiber.*

Pick $b \in B$ and let $U, V$ be neighborhoods of $b$ over which $E$ and $E'$ are trivialized. Thus locally $f$ is represented by a continuous map $\tilde{f} : U \cap f^{-1}(V) \times \mathbb{R}^n \to U \cap f^{-1}(V) \times \mathbb{R}^n$. Define a map $g : U \cap f^{-1}(V) \to GL_n(k)$ by sending $x$ to the invertible transformation $\tilde{f}_x$. It is clear that $g$ is continuous, whence by Cramer's rule, for instance, $g^{-1}(x) := (\tilde{f}_x)^{-1}$ is also continuous. This locally defines an inverse morphism for $f$. These local inverses can be seen to agree on the overlaps, so they can be "glued" together to form a global inverse morphism. We will say more about this in a later discussion on "gluing" constructions for vector bundles.

The proof of this proposition is more subtle than it first seems, but working over the local trivialization is necessary, since replacing the topology on $E$ with the one given locally by $\widehat{B} \times \mathbb{R}^n$, for instance, makes the statement of the theorem false.

**Example 3.2.** There are always **trivial bundles** $B \times \mathbb{R}^n$.

**Example 3.3.** If $M \subset \mathbb{R}^N$ is a smooth $n$-manifold, then $TM$ and $T^*M$ are vector bundles over over $M$. If $f : M \to M$ is a smooth map, then $f_* : TM \to TM$ and $f^* : T^*M \to T^*M$ are morphisms.

We see that the projection map $M \times \mathbb{R}^N \to M$ restricted to $TM$ has fibers isomorphic with $\mathbb{R}^n$. For a system of local coordinates $(U_\alpha, h_\alpha)$, the maps

$$U_\alpha \times \mathbb{R}^n \ni (u, \sum a_i \frac{\partial}{\partial x_i}) \mapsto (h_\alpha(u), \sum a_i \frac{\partial h_\alpha}{\partial x_i}(u))$$

provide a local trivializations for $M$ by the inverse function theorem. A similar argument works for $T^*M$.

**Example 3.4.** For $\mathbb{R}P^n$ there is the **canonical line bundle** $(E_n^1, \gamma_n^1)$ consisting of all $(L, \vec{v}) \in \mathbb{R}P^n \times \mathbb{R}^{n+1}$ with $\vec{v} \in L$. There is the analogous line bundle for $\mathbb{C}P^n$.

**Example 3.5.** For the real Grassmannian $G_k(n)$ there is a generalization of the canonical line bundle $(E_n^k, \gamma_n^k)$ given by all $(V, v)$ with $\vec{v} \in V$.

3.2. **Sections of Vector Bundles.** For $(E, \xi) \in \mathrm{Vect}_k(B)$, a continuous map $s : B \to E$ is a **section** if $\xi(s(b)) = b$ (i.e., $s(b) \in \xi^{-1}(b)$) for all $b \in B$. We can think of sections as a continuously varying family of vectors over $B$. We let $\Gamma(E, \xi)$ denote the family of sections of $(E, \xi)$. We will often shorten this to $\Gamma(E)$ when the map $\xi$ is implicit. Note that there is a linear structure on $\Gamma(E, \xi)$ via fiberwise addition and scalar multiplication. If $E$ is the tangent bundle over $M$, then an element $s \in \Gamma(TM)$ is often referred to as a **vector field**.

The next result is extremely important. Basically, it states that a vector bundle is trivial exactly when there exists a global basis.

**Proposition 3.6.** *An $n$-bundle $(E, \xi)$ is isomorphic to the trivial $n$-bundle $B \times \mathbb{R}^n$ if and only if there exist exactly $n$ sections $s_1, \ldots, s_n \in \Gamma(E, \xi)$ such that $\{s_1(b), \ldots, s_n(b)\}$ span $\xi^{-1}(b)$ for all $b \in B$.*

Clearly, the existence of such a family of sections is invariant under isomorphism, and such a family exists for a trivial $n$-bundle. (Let $(e_i)$ be the canonical basis for $\mathbb{R}^n$, then define $s_i(b) := (b, e_i) \in B \times \mathbb{R}^n$.) On the other hand, the map $f : B \times \mathbb{R}^n \to E$ given by $f : (b, \sum \alpha_i e_i) \mapsto \sum \alpha_i s_i(b)$, again the right sum is understood to be the sum within the fiber, is a well-defined morphism of bundles which is a fiberwise isomorphism. Whence, by Proposition 3.1, $f$ is an isomorphism of bundles.

## 4. Lecture 4

**4.1. Classifying Vector Bundles: First Results.** The goal of this section will be to work through some basic examples, distinguishing trivial from nontrivial bundles. We begin with $TS^n$, the tangent bundle to the $n$-spheres.

**Example 4.1.** $TS^1$ is trivial.

This is easy to see as $s(\cos(\theta), \sin(\theta)) = (-\sin(\theta), \cos(\theta))$ is a nowhere zero section of a line bundle, so Proposition 3.6 applies.

To introduce a bit of notation, we say that a smooth manifold $M$ is **parallelizable** if $TM$ is isomorphic to the trivial $n$-bundle.

**Example 4.2.** $TS^2$ is nontrivial. The tangent bundle to any even-dimensional sphere $TS^{2n}$ is nontrivial.

Suppose that $s_1, \ldots, s_n \in \Gamma(TS^n)$ give a trivialization of $TS^n$. In particular, each $s_i$ is nowhere zero. The map $s_i(x) \mapsto \|s_i(x)\|$, where $\|\cdot\|$ is the euclidean norm on $\mathbb{R}^{n+1}$, is seen to be continuous, whence $\dot{s}_i(x) = \frac{s_i(x)}{\|s_i(x)\|}$ is also continuous. Thus $\dot{s}_i(x) \in S^n$ and $\dot{s}_i(x) \perp x$. We now observe:

**Proposition 4.3.** *If there is a continous map $\nu : S^n \to S^n$ such that $\nu(x) \perp x$ for all $x \in S^n$, then the antpodal map is homotopic to the identity.*

The required homotopy is given by $h_t(x) := \cos(\pi t)x + \sin(\pi t)\nu(x)$ for $t \in [0, 1]$. For $S^1$ note that this homotopy just rotates the circle through $\pi$ radians. Since we have shown in Lecture 2 that the antipodal map is not homotopic to the identity in even dimensions, we have shown by contradiction that $TS^{2n}$ is not trivializable. In fact, we have shown much more:

**Proposition 4.4.** *There is no vector field $s \in \Gamma(TS^{2n})$ which is everywhere nonzero.*

This result is colloquially known as the "Hairy Ball Theorem".

With $TS^1$ and $TS^{2n}$ squared away for the moment, we turn to the tangent bundles of higher dimensional odd spheres. Classification here becomes much more nuanced. We first observe that $TS^{2n-1}$ always admits a nowhere zero vector field given by

$$s(x_0, x_1, \ldots, x_{2n-1}) = (-x_1, x_0, -x_3, x_2, \ldots, -x_{2n-1}, x_{2n-2}).$$

**Example 4.5.** $TS^3$ is parallelizable.

To see that $TS^3$ is parallelizable, we will explicitly construct a global basis. To do so, identify $\mathbb{R}^4$ with the quaternions $\mathbb{H}$, sending the standard orthonomal basis to $\{1, i, j, k\}$. This isometrically identifies the euclidean norm with the modulus. The units in $\mathbb{H}$ are thus canonically identified with $S^3$. Since (right) multiplication by a unit leaves the modulus unchanged, we can view right multiplication by a unit as an orthogonal transformation of $\mathbb{R}^4$ (hint: use the polarization identity for inner products). Thus for any $z \in S^3$, $\{z, iz, jz, zk\}$ is again an orthonormal basis. Thus $s_\epsilon(z) := \epsilon z$ for $\epsilon \in \{i, j, k\}$ defines a family of three fiberwise linearly independent sections of $TS^3$ as $s_\epsilon(z) \perp z$ for all $z \in S^3$. Hence, $TS^3$ is isomorphic to the trivial 3-bundle over $S^3$. It is an exercise to write these sections in terms of coordinates in $\mathbb{R}^4$.

**Example 4.6.** $TS^7$ is parallelizable.

This follows in more or less the same way as for $TS^3$ using the division algebra structure on the octonions $\mathbb{O}$. We have thus exhausted all real division algebras, and in fact all parallelizable spheres! We will see later that the two are very much related.

**Example 4.7.** The $n$**-torus** $\mathbb{T}^n$ is the $n$-fold direct product of $S^1$. The coordinate embeddings smoothly realize $\mathbb{T}^n$ in $\mathbb{R}^{2n}$. $\mathbb{T}^n$ is parallelizable.

The tangent bundle $T\mathbb{T}^n$ is isomorphic in the obvious way to the $n$-fold product of the bundles $TS^1$, each of which is trivializable.

**Example 4.8.** Let $\Sigma_g$ be a compact orientable surface of genus $g$. Then $\Sigma_g$ is not parallelizable unless $g = 1$, i.e., the euler characteristic $\chi(\Sigma_g) = 2 - 2g = 0$.

This is a nontrivial result. Let $s \in \Gamma(T\Sigma_g)$ be a vector field with (finitely many) isolated zeroes. Let $x \in \Sigma_g$ be a point where a zero occurs. Pick a small closed disc around $x$ (one that lies inside a chart in some atlas) on which no other zero occurs. By normalizing $s$ on $\partial D \cong S^1$, we obtain a continuous map $g_x : S^1 \to S^1$ known as the **gauss map**. We define the **index** $\mathrm{ind}_x(s)$ of the vector field at $x$ to be the winding number of the gauss map $g_x$. (In higher dimensions this is the element in $\mathbb{Z} \cong \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$ given by $(g_x)_* : H_n(S^n; \mathbb{Z}) \to H_n(S^n; \mathbb{Z})$.) The **Poincaré–Hopf theorem** states that $\sum_x \mathrm{ind}_x(s) = \chi(\Sigma_g)$ where the sum is understood to be over all zeroes of the vector field. With a little more work, this implies that there is no everywhere nonzero vector field on any compact orientable surface with nonzero euler characteristic.

**Example 4.9.** A Lie group $G$ is parallelizable.

Pick a basis $v_1, \dots, v_n \in TG_e$. Define vector fields $s_i(g) := (L_g)_* v_i$ for $i = 1, \dots, n$. Note that $(L_{g^{-1}})_* \circ (L_g)_* = (L_e)_* = \mathrm{id}_{TG_e}$, from which it follows that $s_1, \dots, s_n$ form a global basis for $TG$.

As a consequence, no even-dimensional sphere can be given the structure of a Lie group. In fact, $S^1$ and $S^3$ are the only spheres which admit a Lie group structure. We have that $S^1 \cong U(1)$ and $S^3 \cong SU(2)$ since an element $A \in U(2)$ with $\det(A) = 1$ is of the form $A = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$ with $|\alpha|^2 + |\beta|^2 = 1$.

**Example 4.10.** Take the quotient of $[-1, 1] \times \mathbb{R}$ given by the identification $(1, t) \sim (-1, -t)$. This produces a line bundle $(E, \mu)$ over $S^1$ known as the **Möbius bundle**.

**Proposition 4.11.** *The Möbius bundle on* $S^1 \cong \mathbb{RP}^1$ *is isomorphic to the canonical line bundle* $\gamma^1$.

Define a map $\Theta : [-1, 1] \times \mathbb{R} \to E^1$ as follows. If $x \in [0, 1]$ map $(x, t) \mapsto (e^{i\pi t}, te^{i\pi t})$, and if $x \in [-1, 0]$, map $(x, t) \mapsto (e^{i\pi t}, t\overline{e^{i\pi t}})$. This passes to a bundle isomorphism $\theta : (E, \mu) \to (E^1, \gamma^1)$. Note we are identifying $\mathbb{R}^2$ and $\mathbb{C}$ without any attendant shame.

Similarly, the canonical line bundle $\gamma_n^1$ on $\mathbb{RP}^n$ can be seen as the quotient of the trivial bundle line bundle on $S^n$ by $(x, t) \sim (-x, -t)$. Realizing the trivial line bundle as the normal bundle $NS^n$ by $(x, t) \mapsto (x, tx)$ gives $\gamma_n^1$ as the quotient of $NS^n$ under $(x, tx) \sim (-x, tx)$. It is easy to see that the quotient of $NS^n$ under $(x, tx) \sim (-x, -tx)$ gives the trivial line bundle on $\mathbb{RP}^n$.

**Proposition 4.12.** *For each $n$, the canonical line bundle $(E_n^1, \gamma_n^1)$ is not trivializable.*

For line bundles trivializabilty is equivalent to the existence of a nowhere zero section. Let $s \in \Gamma(\gamma_n^1)$ be a section. Precomposing with the quotient map $S^n \to \mathbb{R}P^n$, we have a maps $\tilde{s} : S^n \to E_n^1$, so $\tilde{s}(x) = (x, t(x)x)$ for some continuous map $t : S^n \to \mathbb{R}$. Since $\tilde{s}$ facotrs through $\mathbb{R}P^n$ we have

$$\tilde{s}(-x) = (-x, t(-x) \cdot (-x)) = (-x, -t(-x)x) = (x, t(x)x) = \tilde{s}(x)$$

so $t(-x) = -t(x)$. Every continuous, odd function on the sphere must achieve zero somewhere by the intermediate value theorem, whence $\gamma_n^1$ admits no everywhere nonzero section.

## 5. Lecture 5

5.1. **Subbundles of Trivial Bundles and Frames.** Before discussing abstract constructions (direct sums, (alternating) tensor products, etc.) over vector bundles, we pause to look at the more concrete notions of sum and complementation when two bundles are sitting within the common "frame" of a larger, trivial bundle.

Let $(E, \xi) \in \mathrm{Vect}_{\mathbb{R}}(B)$ be a subbundle of the trivial bundle $B \times \mathbb{R}^N$. We will say that $(E, \xi)$ is **complemented** in $B \times \mathbb{R}^N$ by the subbundle $(F, \eta)$ if the fibers of $F$ at each point is the orthogonal complement of the fiber of $E$, i.e., $F_b = E_b^{\perp}$ for all $b \in B$. We will say that $(E, \xi)$ and $(F, \eta)$ form an **internal direct sum decomposition** of $B \times \mathbb{R}^N$ if $E$ and $F$ fiberwise span $\mathbb{R}^N$.

In fact any subset any $F \subset B \times \mathbb{R}^N$ which complements a subbundle in this way is automatically a subbundle.

**Proposition 5.1.** *Let $E \subset B \times \mathbb{R}^N$ be a subbundle. Setting $E^{\perp}$ to be the fiberwise orthogonal complement of $E$, then $E^{\perp}$ is also a subbundle.*

Let $P_b$ be the orthogonal projection from $\mathbb{R}^N$ onto $E_b$. Since $E$ is a subbundle, the map $B \ni b \mapsto P_b \in \mathrm{Hom}(\mathbb{R}^N, \mathbb{R}^N)$ is continuous, whence for any $b \in B$ there is an open neighborhood $U$ and a continuous map $\varphi : U \to GL_N(\mathbb{R})$ such that $P_x = \varphi_x^{-1} P_b \varphi_x$ for all $x \in U$. In fact, one can take $\varphi : U \to O(N)$, without loss of generality. The map $\psi(x) := \varphi(x) P_b$ gives a fiberwise linear homeomorphism $U \times \mathbb{R}^n \cong E|_{U \times \mathbb{R}^N}$ where $n = \dim(P_b)$. Likewise $\psi^{\perp}(x) := \varphi(x) P_b^{\perp}$ gives a fiberwise linear homeomorphism $U \times \mathbb{R}^{N-n} \cong E^{\perp}|_{U \times \mathbb{R}^N}$ which shows that $E^{\perp}$ is also a subbundle.

**Proposition 5.2.** *If $E$ and $F$ are subbundles of $B \times \mathbb{R}^N$ which form an internal direct sum decomposition, then $F$ is isomorphic to $E^{\perp}$.*

By the Gram-Schmidt process there is a continuous map $\varphi : B \to GL_N(\mathbb{R})$ so that $\varphi_b$ maps $F_b$ isomorphically onto $E_b^{\perp}$.

**Example 5.3.** For a smooth manifold $M \subset \mathbb{R}^N$, the tangent and normal bundles complement each other in $M \times \mathbb{R}^N$.

**Example 5.4.** The map $\theta$ defined immediately after Proposition 4.11 and a copy of it given by rotating the circle $\pi/2$ radians give two complemented copies of the Möbius bundle in $S^1 \times \mathbb{R}^2$.

**Example 5.5.** Generalizing the previous example, The quotient of $S^n \times \mathbb{R}^n$ under the relation $(x, v) \sim (-x, -v)$ decomposed as an internal direct sum of $n+1$ copies of the canonical line bundle $\gamma_n^1$. It way also be seen to decompose as the sum of the tangent and normal bundles over $\mathbb{R}P^n$.

Given a bundle $(E, \xi) \in \mathrm{Vect}_{\mathbb{R}}(B)$ a **frame** is an embedding of $E$ into a trivial bundle $B \times \mathbb{R}^N$. There is an alternate description on frames given by essentially by the proof of Proposition 3.6

**Proposition 5.6.** *A vector bundle $(E, \xi)$ embeds in $B \times \mathbb{R}^N$ if and only if there are sections $s_1, \ldots, s_N \in \Gamma(E, \xi)$ such that $\{s_1(b), \ldots, s_N(b)\}$ spanes the fiber at $b$ for all $b \in B$.*

**Proposition 5.7.** *If $B$ is compact, then every vector bundle over $B$ admits a frame.*

Let $(E, \xi) \in \mathrm{Vect}_{\mathbb{R}}(B)$. Since $B$ is compact, we can pick a finite collection $(U_i, \varphi_i)_{i=1}^n$ of local trivializations $\varphi_i : E_{U_i} \to U_i \times \mathbb{R}^m$ of $(E, \xi)$ along with a subordinate partition of unity $\{p_1, \ldots, p_n\}$ (we can repeat elements of the cover to so that the indices match). We define maps $\tilde{\varphi}_i : E \to B \times \mathbb{R}^m$ by $\tilde{\varphi}_i(x) = (x, p_i(x)\varphi_i(x))$ if $x \in U_i$ and $\tilde{\varphi}_i(x) = (x, 0)$ otherwise. One can check that the image $\tilde{\varphi} := \times_{i=1}^n \tilde{\varphi}_i : E \to B \times \mathbb{R}^{mn}$ is a subbundle and that $\varphi$ is an isomorphism of $E$ with its image.

**Exercise 5.8.** The Möbius bundle admits two local trivializations on the open sets obtained by removing the north or south pole of $S^1$. Use this to construct an explicit frame embedding for the Möbius bundle in $S^1 \times \mathbb{R}^2$.

## 6. Lecture 6

We conclude with a few remarks expanding on what is implicit in the constructions above. To set some notation, let $\mathrm{Proj}(\mathbb{R}^N) \subset \mathrm{Hom}(\mathbb{R}^N, \mathbb{R}^N) \cong M_N(\mathbb{R})$ be the set of all orthogonal projections. First,

**Proposition 6.1.** *There is a bijective correspondence between $n$-subbundles of $B \times \mathbb{R}^N$ and continuous maps $f : B \to \mathrm{Proj}(\mathbb{R}^N)$ where the image of each point has rank $n$.*

For a subbundle $E \subset B \times \mathbb{R}^N$ the map is obtained by mapping $b$ to the orthogonal projection onto its fiber. In the reverse direction, consider the subset $E_f := \{(b, v) : f_b(v) = v\}$. For each $b \in B$ there is an open neighborhood $U$ so that for all $x \in U$, $f_x$ and $f_b$ are sufficiently close orthogonal projections, thus there is a canonical $\varphi_x \in GL_N(\mathbb{R})$ which conjugates the two, $f_x = \varphi_x^{-1} f_b \varphi_x$ and varies continuously in $x$. Thus identifying $\mathbb{R}^n$ with the fiber at $b$, $\tilde{\varphi} : U \times \mathbb{R}^n \to E_f|_U$ given by $\tilde{\varphi}(x, v) := (x, \varphi_x v)$ is a local trivialization. Thus $E_f$ is a subbundle.

This observation can be rephrased slightly:

**Proposition 6.2.** *For a compact space $B$, there is a bijective correspondence between subbundles of $B \times \mathbb{R}^N$ and elements $p \in M_N(C_{\mathbb{R}}(B))$ satisfying $p = p^t = p^2$, where $C_{\mathbb{R}}(B)$ is the ring of continuous real-valued funcitons on $B$.*

This follows by the algebra isomorphism $M_N(C_{\mathbb{R}}(B)) \cong C(X, M_N(\mathbb{R}))$ where $C(X, Y)$ denotes the space of continuous maps from $X$ to $Y$.

The set $\mathcal{E}_n^N$ of all $n$-subbundles of $B \times \mathbb{R}^N$ can be topologized by, for instance, taking the supremum of the hausdorff distances between unit balls in the fibers of the bundles over all points $b \in B$. There are many other ways to describe this topology such as by the subspace topology on the contiuous maps in $C(B, M_N(\mathbb{R}))$ whose image lies in the rank $n$ projections.

**Proposition 6.3.** *Let $B$ be compact. For any $E \in \mathcal{E}_n^N$ there is an open neighborhood $\mathcal{U}$ such that $E$ is isomorphic to $F$ for all $F \in \mathcal{U}$.*

In particular this implies that all vector bundles belonging to any single connected component of $\mathcal{E}_n^N$ are in the same isomorphism class. However, $N$ may not be sufficiently large relative to $n$ for the converse to hold, i.e., vector bundles in two distinct connected components may be isomorphic. We will discuss these issues in detail later. The point here is that while it seems like there is a vast continuum of subbundles of a trivial bundle, they actually come in discretely many flavors. This is the first indication as to why the study of vector bundles belongs in the realm of algebraic topology and homotopy theory.

To see why this holds, note that if $F$ is an $n$-subbundle sufficiently close to $E$, then their unit balls in each of the fibers are uniformly close. This means there is some neighborhood $V$ of the identity in $GL_N(\mathbb{R})$ and a continuous map $f : B \to V$ such that $f_b E_b = F_b$ for all $b \in B$, whence $E$ and $F$ are isomorphic.

6.1. **Inner Products.** While not every vector bundle over a paracompact space admits a frame, we have, in a sense, the next best thing.

For a vector bundle $(E, \xi) \in \mathrm{Vect}_{\mathbf{k}}(B)$, define $E \times_B E := \{(x, y) \in E \times E : \xi(x) = \xi(y)\}$. The projection map $\tilde{\xi} : E \times_B E \to B$ is the restriction of $\xi \times \xi$. An **inner product** for $(E, \xi)$ is a map $\langle \cdot, \cdot \rangle : E \times_B E \to \mathbf{k}$ which restricts to a nondegenerate inner product over each fiber.

**Proposition 6.4.** *If $B$ is paracompact, then any vector bundle over $B$ admits an inner product.*

Let $\{(U_i, \varphi_i)\}_{i \in I}$ be a local trivialization for $(E, \xi)$ which is locally finite. Pick a partition of unity $\{p_i\}_{i \in I}$ subordinate to this cover with $\mathrm{supp}(p_i) \subset U_i$. (By repeating an element of the cover multiple times we can assume the cover and partition are indexed by the same set.) The trivial bundle $U_i \times \mathbf{k}^n$ admits an inner product $\langle \cdot, \cdot \rangle (x, v, w) := \langle v, w \rangle$. This defines an inner product $\langle \cdot, \cdot \rangle_i$ on $E|_{U_i}$ by precomposition with $\varphi_i$. We can then define a inner product on $E$ by

$$\langle \cdot, \cdot \rangle := \sum_{i \in I} \langle \cdot, \cdot \rangle_i \, (p_i \circ \tilde{\xi}).$$

## 7. Lecture 7

7.1. **The Gluing Construction and Cocycles.** Let $(E, \xi) \in \mathrm{Vect}_{\mathbf{k}}(B)$ be a vector bundle with fixed local trivialization $\{(U_i, \varphi_i)\}_{i \in I}$. The maps $\varphi_i \circ \varphi_j^{-1} : (U_i \cap U_j) \times \mathbf{k}^n \to (U_i \cap U_j) \times \mathbf{k}^n$ define maps $g_{ij} : U_i \cap U_j \to GL_n(\mathbf{k})$ by

$$\varphi_i \circ \varphi_j^{-1}(x, v) = (x, g_{ij}(x)v).$$

By this identity, the maps $g_{ij}$ satisfy the identity

$$g_{ij}g_{jk} = g_{ik}$$

when restricted to the common domain $U_i \cap U_j \cap U_k$.

For a topological space $B$ equipped with an open covering $\{U_i\}_{i\in I}$, we will call any such set of maps $g_{ij} : U_i \cap U_j \to GL_n(\mathbf{k})$ a **cocycle** over $B$. You are probably wondering at this moment whether every cocycle comes in a natural way from a vector bundle over $B$: the answer is 'yes.'

**Proposition 7.1** (Gluing Construction). *For every cocycle $\{\{U_i\}, \{g_i j\}\}$ over $B$ there is a vector bundle $(E, \xi)$ with local trivializations $\{(U_i, \varphi_i)\}$ such that $g_{ij} = \varphi_i \circ \varphi_j^{-1}|_{U_i\cap U_j}$ for all $i, j \in I$.*

Let $X := \coprod_{i\in I} U_i$ so that $F := \coprod_{i\in I} U_i \times \mathbf{k}^n$ is a trivial bundle over $X$ with projection map $\pi$. Define an equivalence relation on $F$ by $(x, v) \sim (x, g_{ij}(x)v)$ for $(x, v) \in U_j \times \mathbf{k}^n$ and $(x, g_{ij}(x)v) \in U_i \times \mathbf{k}^n$. Writing $E := F/\sim$, then $\pi$ descends to a projection $\pi_\sim : E \to B$. We see that $(E, \pi_\sim)$ is a vector bundle, since the local trivializations $(U_i, \varphi_i)$ are just the liftings of $U_i$ over the quotient. From this it is automatic that $\varphi_i \circ \varphi_j^{-1} = g_{ij}$!

The gluing construction is the "adjoint" of the map which sends a vector bundle (with a specific local trivialization) to its associated cocycle. To be precise:

**Proposition 7.2.** *let $(E, \xi)$ be a vector bundle, with local trivialization $\{(U_i, \varphi_i)\}$ with associated cocycle $g_{ij}$. Let $(\tilde{E}, \tilde{\xi})$ be the vector bundle obtained from $g_{ij}$ via the gluing construction. Then $E$ and $\tilde{E}$ are isomorphic.*

To see this, note that literally by construction to total map

$$(\coprod \varphi_i)_\sim : E \to \coprod (U_i \times \mathbf{k}^n) \to \tilde{E}$$

is well-defined and a linear isomorphism on each fiber, thus is an isomorphism.

Note that if we have two cocycles $\{\{U_i\}, \{g_{ij}\}\}$ and $\{\{V_i\}, \{h_{ij}\}\}$, by taking the common refinement of the two open covers, and restriciting the maps appropriately, we may assume that both cocycles are defined over a common open cover of $B$.

The main question then becomes, "Given two cocycles $g_{ij}$ and $h_{ij}$ over a common open cover $\{U_i\}$ of $B$, how can we tell whether the associated vector bundles are isomorphic?" We say that a cocycle is **trivial** if there are maps $\lambda_i : U_i \to GL_n(\mathbf{k})$ so that $g_{ij} = \lambda_i\lambda_j^{-1}|_{U_i\cap U_j}$. More generally, we will say that the cocycles $g_{ij}$ and $h_{ij}$ are **equivalent or cohomologous** if $g_{ij} = \lambda_i h_{ij} \lambda_j^{-1}$ for maps $\lambda_i$ as before.

**Proposition 7.3.** *If $\{\{U_i\}, \{g_{ij}\}\}$ and $\{\{U_i\}, \{h_{ij}\}\}$ are cocycles over $B$, then they are equivalent if and only if the induced vector bundles are equivalent. To put it another way, two vector bundles are isomorphic if and only if they have equivalent cocycles over some (any) common locally trivializing cover.*

Let $(E, \xi), (F, \eta) \in \mathrm{Vect}_{\mathbf{k}}(B)$ and choose a common locally trivializing cover $\{U_i\}_{i\in I}$ with respective local trivializations $\{(U_i, \varphi_i)\}$ and $\{(U_i, \psi_i)\}$. Suppose we have an isomorphism $f : E \to F$. Then the map

$$U_i \times \mathbf{k}^n \xrightarrow{\varphi_i^{-1}} E|_{U_i} \xrightarrow{f} F|_{U_i} \xrightarrow{\psi_i} U_i \times \mathbf{k}^n$$

is a fiberwise isomorphism, whence induces a map $\lambda_i : U_i \to GL_n(\mathbf{k})$. It is easy to check that $g^E_{ij} = \lambda_i g^F_{ij} \lambda_j^{-1}$. Conversely, suppose that $g^E_{ij}$ and $g^F_{ij}$ are equivalent so that $g^E_{ij} = \lambda_i^{-1} g^F_{ij} \lambda_j$. Define a map $\sigma : \coprod(U_i \times \mathbf{k}^n) \to \coprod(U_i \times \mathbf{k}^n)$ by

$$\sigma : U_i \times \mathbf{k}^n \ni (x, v) \mapsto (x, \lambda_i(x)v) \in U_i \times \mathbf{k}^n.$$

The $\sigma$ descends to a well-defined, fiberwise isomorphic map $\sigma_\sim : \coprod(U_i \times \mathbf{k}^n)/\sim g^E_{ij} \to \coprod(U_i \times \mathbf{k}^n)/\sim g^F_{ij}$. As previously discussed these two bundles are isomorphic to $E$ and $F$ respectively; thus, $E$ and $F$ are isomorphic.

By way of an example, let $\{U_1, U_2\}$ be the open cover of $S^1$ the the open sets obtained by removing the second and fourth quarters of the circle, respectively. Then $U_1 \cap U_2 = (0, \pi/2) \cup (\pi, 3\pi/2)$. Since this set is the union of two connected sets, up to equivalence, any cocycle $g_{12} : U_1 \cap U_2 \to GL_1(\mathbb{R})$ assigns a value of either 1 or $-1$ to each interval. If the signs are the same, then the bundle is trivial. In the case $(-1, -1)$ the gluing construction creates a "strip with two twists." If the signs are opposite, then the associated bundle is the Möbius bundle.

## 8. Lecture 8

8.1. **Sums, Tensors, and Other Categorical Constructions.** We know that there are many categorical constructions for vector spaces – direct sums, tensor products, dual spaces, conjugate spaces, alternating tensor products, etc. Thinking of all $\mathbf{k}$-vector spaces as $\mathrm{Vect}_{\mathbf{k}}(\mathrm{pt})$ ("pt" denotes – you guessed it – the one point space) we wish to extend these constructions to $\mathrm{Vect}_{\mathbf{k}}(B)$ for $B$ an arbitrary topological space. Of course, if we wanted to take, say, the dual bundle $(E^*, \xi^*)$ to $(E, \xi)$, then it is logical that fiberwise this should be $E^* = \bigcup_{x\in B}(E_x)^*$. The tricky issue is exactly what topology to assign to give a natural bundle structure. Again, by way of example, if $f : E \to F$ was a bundle morphism, then the dual map $f^* : F^* \to E^*$ defined fiberwise should again be a bundle morphism. The goal here is to show that the gluing construction provides a correct and unifying framework for understanding these constructions.

Let $\tau$ be a (perhaps contravariant) functor from $\mathrm{Vect}_{\mathbf{k}}(\mathrm{pt})$ to $\mathrm{Vect}_{\mathbf{k}'}(\mathrm{pt})$. We will say that $\tau$ is **continuous** if $f \mapsto \tau(f)$ is continuous from $\mathrm{Hom}(V, W)$ to $\mathrm{Hom}(\tau(V), \tau(W))$ (or $\mathrm{Hom}(\tau(W), \tau(V))$ in the contravariant case). For a vector bundle $(E, \xi)$, let $\{\{U_i\}, \{g_{ij}\}\}$ be a cocycle for which the gluing construction yields an isomorphic copy of $(E, \xi)$. The vector bundle $(\tau(E), \tau(\xi))$ is then given by $\coprod(U_i \times \tau(\mathbf{k}^n))/\sim \tau(g_{ij})$, i.e., the bundle induced by the cocycle $\tau(g_{ij})$ which is still continuous since $\tau$ is continuous. This construction preserves the relation of triviality and two cocycles with a common refinement yield isomorphic constructions, from which it follows that this construction is well-defined up to isomorphism. Moreover, if $f : E \to F$ is a bundle morphism, then the fiberwise defined map $\tau(f) : \tau(E) \to \tau(F)$ (or $\tau(f) : \tau(F) \to \tau(E)$ in the contravariant case) is by construction a bundle map.

Here are some examples:

(1) the dual bundle $E^*$.
(2) the **complexification** $E^{\mathbb{C}}$ of a real bundle $E$.
(3) the **conjugate bundle** $\overline{E}$ of a complex bundle. As a reminder the conjugate vector space $\overline{V}$ has the same additive structure as $V$, but $\overline{\lambda v} = \bar{\lambda} \cdot \bar{v}$.

(4) the k-alternating tensor power $\bigwedge^k E$.

Direct sums and tensor products are handled similarly. For $(E,\xi),(F,\eta)\in \mathrm{Vect}_{\mathbf{k}}(B)$, the **direct sum or Whitney sum** $(E\oplus F,\xi\oplus\eta)$ is given by taking the direct sum of associate cocycles over a common open cover, i.e., $g_{ij}^{E\oplus F}=g_{ij}^{E}\oplus g_{ij}^{F}$. The **tensor product** is given by the cocycle $g_{ij}^{E\otimes F}=g_{ij}^{E}\otimes g_{ij}^{F}$.

**Example 8.1.** For $(E,\xi),(F,\eta)\in \mathrm{Vect}_{\mathbf{k}}(B)$, the bundle $\mathrm{Hom}(E,F)$ can be identified with $E^*\otimes F$.

**Exercise 8.2.** Show that if $E\in \mathrm{Vect}_{\mathbb{R}}(B)$ is a line bundle, then $E\otimes E\cong B\times\mathbb{R}$ and if $E\in\mathrm{Vect}_{\mathbb{C}}(B)$ then $E\otimes\overline{E}\cong B\times\mathbb{C}$.

8.2. **Pullbacks.** An easy way to construct a vector bundle over a space is to borrow (or steal) one from another space. As we will see later, the Grassmannians will be particularly generous benefactors. Given topological spaces $B, A$, a continuous map $f: B\to A$, and a vector bundle $(E,\xi)\in\mathrm{Vect}_{\mathbf{k}}(A)$, the **pullback** of $(E,\xi)$ along $f$, denoted as $(f^*E, f^*\xi)$, is the vector bundle fitting into the following commutative diagram:

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "f^*\xi"'] & E \arrow[d, "\xi"'] \\ B \arrow[r, "f"] & A \end{tikzcd}$$

Such an object can be constructed quite naturally as $f^*E=\{(x,b)\in E\times B:\xi(x)=f(b)\}$. The map $f^*\xi$ is just the restriction of the projection $\pi_B: E\times B\to B$ and the top map is the restriction of $\pi_E$. Why is this a vector bundle? Picking a cocycle $\{\{U_i\},\{g_{ij}\}\}$ representing $E$, then $f^*E$ can be identified with the bundle given by the cocycle $\{\{f^{-1}U_i\},\{g_{ij}\circ f\}\}$.

**Exercise 8.3.** For $B\subset A$, the restriction $E|_B$ of a vector bundle $E\in\mathrm{Vect}_{\mathbf{k}}(A)$ can be identified with the pullback of $E$ along the canonical inclusion $B\hookrightarrow A$.

**Exercise 8.4.** For $(E,\xi),(F,\eta)\in\mathrm{Vect}_{\mathbf{k}}(B)$ show that $E\oplus F\cong\xi^*F\cong\eta^*E$.

**Exercise 8.5.** Show that the pullback of the Möbius bundle over the map $f(\theta)=2\theta$ yields the trivial line bundle over $S^1$. Hint: pick a cocycle for the Möbius bundle and compute the new cocycle on $S^1$ under the pullback.

## 9. Lecture 9

9.1. **Structure Groups.** Let $G$ be a topological group. Suppose for a space $B$ admitting some open cover $\{U_i\}$ we have a set of continuous maps $g_{ij}: U_i\cap U_j\to G$ satisfying the cocycle condition $g_{ij}(x)g_{jk}(x)=g_{ik}(x)$ for all $x\in U_i\cap U_j\cap U_k$. We will then say that $G$ is the **structure group** of the cocycle $\{g_{ij}\}$. So to rephrase things slightly (and to sweep a few details under the carpet until later) we can think of a vector bundle as cocycle with structure group $GL_n(\mathbf{k})$. Since, working over $GL_n(\mathbf{k})$ can be unwieldy in may situations, we would like to replace any $GL_n(\mathbf{k})$-valued cocycle with an equivalent one in a smaller, more manageable group. It turns out that we can always do so:

**Proposition 9.1.** *Any real (resp., complex) vector bundle $(E, \xi)$ over a paracompact space $B$ admits a cocycle with structure group $O(n)$ (resp., $U(n)$).*

By Proposition 6.4 there is an inner product on $E$. Let $\{U_i\}$ be a cover over which there is a local trivialization for $E$. For the local trivialization on $U_i$ this inner product induces an inner product $\langle \cdot, \cdot \rangle_x$ on $\mathbf{k}^n$ for each $x \in U_i$, whence $\langle \cdot, \cdot \rangle_x = \langle A_x \cdot, \cdot \rangle$ for some nondegenerate positive definite matrix $A_x$ depending continuously on $x$. Setting $\lambda_i(x) = A_x^{1/2}$ we see that $\lambda_i g_{ij} \lambda_j^{-1}$ fiberwise preserves the standard inner prodict on $\mathbf{k}^n$. Thus $g_{ij}$ is equivalent to a cocycle with structure group $O(n)$ in the real case and $U(n)$ in the complex case.

9.2. **The Bundle Homotopy Theorem.** We have seen in Lecture 6 that if two $n$-vector bundles $E, F \subset B \times \mathbf{k}^N$ belonging to the same path component in the the the space of all $n$-subbundles of $B \times \mathbf{k}^N$, then $E$ and $F$ are isomorphic. The Bundle Homotopy Theorem is a generalization of this result. Before stating it, we give a preliminary lemma.

**Lemma 9.2.** *Let $B$ be a paracompact, hausdorff space and $A \subset B$ be closed. If $(E, \xi) \in \mathrm{Vect}_{\mathbf{k}}(B)$, then any section $s \in \Gamma(E|_A)$ extends to a section $\tilde{s} \in \Gamma(E)$.*

Pick a cover $\{U_i\}$ of $B$ by local trivializations of $E$. Then $s|_{A \cap U_i}$ can be viewed as a vector-valued map on $A \cap U_i$, whence by the Tietze extension theorem (recall paracompact spaces are normal) extends to a section $\tilde{s}_i \in \Gamma(E|_{U_i})$. Choosing a partition of unity $\{p_i\}$ subordinate to the cover, we can then define $\tilde{s}(x) := \sum_i p_i(x) \tilde{s}_i(x)$.

**Proposition 9.3** (Bundle Homotopy Theorem)**.** *Let $B$ be a (para)compact, hausdorff space, $A$ a topological space and $f, g : B \to A$ homotopic maps. Then for any $(E, \xi) \in \mathrm{Vect}_{\mathbf{k}}(A)$, $f^*(E) \cong g^*(E)$.*

We will assume that $B$ is compact. Let $h : B \times [0, 1] \to A$ be the homotopy from $f$ to $g$. We have the pullback $h^*(E)$ and we define $E_t \in \mathrm{Vect}_{\mathbf{k}}(B)$ to be $h^*(E)$ restricted to $B \times \{t\}$. We will show that if $E_t \cong F$, then there exists $\epsilon > 0$ such that $E_s \cong F$ for all $s \in (t - \epsilon, t + \epsilon)$. The result will then follow by the connectedness of $[0, 1]$.

We first use the pullback under the projection $\pi_B : B \times [0, 1] \to B$ to extend $F$ to a vector bundle $\pi_B^* F$ over $B \times [0, 1]$. Note the restriction of $\pi_B^* F$ to $B \times \{t\}$ is naturally isomorphic to $F$ for all $t$. Since $E_t$ and $F$ are isomorphic there is a section $\sigma : B \to \mathrm{Hom}(E_t, F)$ such that $\sigma(b)$ is invertible for all $b \in B$. Identifying $B$ with $B \times \{t\}$ and $\mathrm{Hom}(E_t, F)$ with $\mathrm{Hom}(h^*(E), \pi_B^* F)$ restricted to $B \times \{t\}$, by the previous lemma $s$ extends to a section $\tilde{\sigma} : B \times [0, 1] \to \mathrm{Hom}(h^*(E), \pi_B^* F)$. Since the $GL_n(\mathbf{k})$ is open in $\mathrm{Hom}(\mathbf{k}^n, \mathbf{k}^n)$ it follows that for all $b \in B$ there is an open neighborhood of $(b, t)$ in $B \times [0, 1]$ where $\tilde{\sigma}$ takes values in the invertibles. Passing to a finite subcover there is a strip $B \times [t\epsilon, t + \epsilon]$ on which $\tilde{\sigma}$ takes values in the invertibles. This means that for all $s \in [t - \epsilon, t + \epsilon]$ there is an isomorphism of $E_s$ and $\pi_B^* F|_{B \times \{s\}} \cong F$.

It is an exercise to adapt this proof to the paracompact setting. Truthfully, our proof actually suggests the following convenient restatement, which can be seen to be equivalent:

**Proposition 9.4** (Bundle Homotopy Theorem, general version)**.** *Let $B$ be a paracompact, hausdorff space. For any vector bundle $E$ over $B \times [0, 1]$ the vector bundles $E_0, E_1 \in \mathrm{Vect}_{\mathbf{k}}(B)$ obtained by restricting to $B \times \{0\}$ and $B \times \{1\}$ are isomorphic.*

Here is one of the main corollaries of the Bundle Homotopy Theorem:

**Proposition 9.5.** *If* $B$ *is paracompact, hausdorff, and contractible, then every vector bundle over* $B$ *is trivial.*

Choose a point $x \in B$. Since $B$ is contractible the identity map id is homotopic to the constant map $c_x : B \to \{x\}$. For $(E, \xi) \in \mathrm{Vect}_{\mathbf{k}}(B)$, $E \cong \mathrm{id}^*(E)$ while on the other hand $c_x^*(E)$ is trivial.

For a paracompact, hausdorff space $X$ we define the **cone** $CX$ over $X$ to be the quotient of $X \times [0, 1]$ obtained by identifying $X \times \{1\}$ to a single point. Note that $CX$ is contractible, whence all vector bundles over $CX$ are trivial.

## 10. Lecture 10

The Bundle Homotopy Theorem gives strong evidence that the study of vector bundles properly belongs to the realm of algebraic topology. The main goal of the next several lectures will be to develop this idea in precise detail.

10.1. **Clutching.** We will focus on one straightforward but profound consequence of the Bundle Homotopy Theorem:

**Proposition 10.1.** *Let* $B$ *be a paracompact, hausdorff space. Suppose that* $B$ *is* **locally contractible**, *i.e.,* $B$ *admits an open cover* $\{U_i\}$ *where each* $U_i$ *is contractible. Then every vector bundle over* $B$ *can be obtained from the gluing construction for some cocycle* $\{g_{ij}\}$ *defined over* $\{U_i\}$.

We first apply this over the spheres where it is known as **clutching**. Specifically we write $S^n = D^n_+ \cup_{S^{n-1}} D^n_-$ where $D^n_\pm$ are the northern and southern hemispheres (topologically both are the (closed) $n$-disk) and $S^{n-1} = D^n_+ \cap D^n_-$ is the equator. The previous proposition tells us the any $N$-vector bundle over $S^n$ is determined by a single continuous map $g : S^n \to GL_N(\mathbf{k})$ as the cocycle condition is trivially satisfied. Note that technically we should enlarge the hemispheres slightly and speak of the map $g$ as being defined over a band $S^{n-1} \times (-\epsilon, \epsilon)$, but as we will see this works out to the same thing.

[*Here I went over examples of clutching maps for* $TS^2$ *and* $\mathbb{C}P^1$. *There is a strong pictorial component to these arguments which may take some time for me to reproduce here. In the meantime see Hatcher.*]

**Proposition 10.2.** *If* $f, g : S^{n-1} \to GL_N(\mathbf{k})$ *are homotopic then the vector bundles* $E_f, E_g \in \mathrm{Vect}_{\mathbf{k}}(S^n)$ *induced by clutching are isomorphic.*

Choose a homotopy $h : S^{n-1} \times [0, 1] \to GL_N(\mathbf{k})$. Since $\{D^n_+ \times [0, 1], D^n_- \times [0, 1]\}$ gives a cover of $S^n \times [0, 1]$ with intersection $S^{n-1} \times [0, 1]$ we can apply the gluing construction over $h$ to create a vector bundle $E_h$ over $S^{n-1} \times [0, 1]$ whose restrictions to $S^n \times \{0\}$ and $S^n \times \{1\}$ are isomorphic as $S^n$ bundles to $E_f$ and $E_g$, respectively. By the general form of the Bundle Homotopy Theorem $E_f \cong E_g$.

For two topological spaces $X$ and $Y$, we define $[X, Y]$ to be the space of all homotopy classes of maps from $X$ to $Y$ with the quotient compact-open topology from $Y^X$. The previous proposition thus shows there is a map $[S^{n-1}, GL_N(\mathbf{k})] \to \mathrm{Vect}^N_{\mathbf{k}}(S^n)$. In the case that $\mathbf{k} = \mathbb{C}$ we will now show that this is an isomorphism.

**Proposition 10.3.** *There is a bijective correspondence between* $\mathrm{Vect}^N_{\mathbb{C}}(S^n)$ *and* $[S^{n-1}, GL_N(\mathbb{C})]$.

We have already defined the map in one direction, so let's construct the inverse. Since $\{D^n_+, D^n_-\}$ is a contractible cover, any vector bundle over $S^n$ is obtained from some clutching function over $g : S^{n-1} \to GL_N(\mathbb{C})$. Recall that the choice of this clutching function (cocycle) is not well-defined but any other such choice $g' : S^{n-1} \to GL_N(\mathbb{C})$ will be equivalent via maps $\lambda_+ : D^n_+ \to GL_N(\mathbb{C})$ and $\lambda_- : D^n_- \to GL_N(\mathbb{C})$. However, recall that $GL_N(\mathbb{C}$ is path connected, whence as $D^n_\pm$ are contractible $\lambda_\pm$ are homotopic to constant maps to the identity, thus $g' = \lambda_-^{-1} g \lambda_+ \sim g$. This shows that the map from a vector bundle to the homotopy class of its clutching function is a well-defined inverse to the gluing construction.

**Exercise 10.4.** Modify this proof to show there is a bijective correspondence between the oriented, real N-bundles $\mathrm{Vect}^N_{\mathbb{R},+}(S^n)$ and homotopy classes $[S^{n-1}, GL^+_N(\mathbb{R})]$. (Recall $GL^+_N(\mathbb{R})$ is all invertibles with positive determinant.)

In fact there was nothing particularly special about $S^n$ here. For any space $X$, we define the **suspension** $SX$ as the quotient of $X \times [-1, 1]$ given by identifying $X \times \{-1\}$ a single point and $X \times \{1\}$ to a single point. Note that $SX = CX_- \cup_{X \times \{0\}} CX_+$, where $CX_\pm$ are copies of the cone over $X$. When $X = S^{n-1}$ this is nothing other than the hemispherical decomposition we have been working with above. Moreover, since the cones are always contractible, we have that the reasoning above applies verbatim, so:

**Proposition 10.5.** *Let $X$ be a (para)compact, hausdorff space. Then there is a bijective correspondence between* $\mathrm{Vect}^N_{\mathbb{C}}(SX)$ *and* $[X, GL_N(\mathbb{C})]$.

Since for any clutching function $g : X \to GL_N(\mathbb{C})$ we can reduce its structure group to $U(N)$, by using the above identification we have that $[X, GL_N(\mathbb{C})] \cong [X, U(N)]$. However this can be seen much more directly:

**Proposition 10.6.** $U(n)$ *is a deformation retract of* $GL_n(\mathbb{C})$.

Let $A \in GL_n(\mathbb{C})$. By polar decomposition $A = u|A|$ for some $u \in U(n)$. Since $|A|$ is positive and invertible, it can be checked that $t|A| + (1-t)1$ is invertible for all $t \in [0, 1]$. The maps $h_t : A \mapsto tu|A| + (1-t)u$ provide the deformation retract.

Since $U(n)$ is path connected, the space of based homotopies is equivalent to the space of unbased homotopies, whence we have that

$$\mathrm{Vect}^N_{\mathbb{C}}(S^n) \cong [S^{n-1}, U(N)] \cong \pi_{n-1}(U(N)).$$

What are these groups? Much like in the case of homotopy groups of spheres, they are still largely mysterious. However, consider the inductive limit $U := \varinjlim U(n)$ under the connecting maps $i_{m,n} : U(m) \to U(n)$ where for $m < n$

$$i_{m,n} : U(m) \ni A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1_{n-m} \end{pmatrix}.$$

We have that $\pi_k(U) = \varinjlim \pi_k(U(n))$ under the connecting maps $(i_{m,n})_* : \pi_k(U(m)) \to \pi_k(U(n))$. Miraculously, it turns out that $\pi_k(U)$ is computable and in fact $\pi_0(U) = 0$, $\pi_1(U) \cong \mathbb{Z}$, and $\pi_{i+2}(U) \cong \pi_i(U)$! This result is a consequence of the **Bott Periodicity Theorem**.

## 11. Lecture 11

11.1. **Principal $G$-bundles.** [*There was a presentation by Joe Knight on this topic*]

11.2. **Defining $K^{-1}(X)$.** From the clutching construction we have seen that $\mathrm{Vect}^N_{\mathbb{C}}(SX) \cong [X, U(N)]$ If $X$ is compact the inductive limit topology on U is compatible with the quotient compact-open topology on $[X, U(n)]$ whence $\varinjlim [X, U(n)] \cong [X, U]$.

For a compact space $X$, we then define $K^{-1}(X) := [X, U]$.

## 12. Lectures 12

12.1. $K^{-1}(X)$ **as an abelian group.** We will now discuss the abelian group structure on $K^{-1}(X)$. For maps $g_1, g_2 : X \to U$ there are two immediately obvious ways of going about taking a product of $[g_1], [g_2] \in K_{-1}(X)$. First one could use the fact that the range has a group structure to define $[g_1] * [g_2] = [g_1 g_2]$. Secondly, more in keeping with the vector bundle perspective, one could notice that by compactness the ranges of these maps must both be contained in $U(n)$ for some $n$ sufficiently large, so we could define $[g_1] + [g_2] := [g_1 \oplus g_2]$, where

$$(g_1 \oplus g_2)(x) := \begin{pmatrix} g_1(x) & 0 \\ 0 & g_2(x) \end{pmatrix} \in U(2n) \subset U\,.$$

It is a quick exercise to check this operation is well-defined at the level of homotopy classes. By post-composing with a rotation, we see that $g_1 \oplus g_2$ and $g_2 \oplus g_1$ are homotopic, whence $[g_1] + [g_2] = [g_1 \oplus g_2] = [g_2 \oplus g_1] = [g_1] + [g_2]$ which shows that "+" is at least a commutative operation on pairs of homotopy classes.

Let's get back to that first operation we defined. For the block embedding $U(n) \times U(n) \subset U(2n)$, let $\rho_t$ be a path of rotations such that $\rho_0$ is the identity and conjugating by $\rho_1$ interchanges the two copies of $U(n)$. Thus for $g_1, g_2 : X \to U(n)$ we have that $g_1 \rho_t^{-1} g_2 \rho_t : X \to U(2n)$ shows that $[g_1] * [g_2] = [g_1] + [g_2]$. This also establishes associativity of the sum, thus summation gives an abelian monoid structure to $K^{-1}(X)$.

We will now show why $K^{-1}(X)$ is actually a group. First note that for $1_X$, the constant maps to the identity, we have that $[g] + [1_X] = [g \cdot 1_X] = [g]$, whence $[1_X]$ is the identity. The natural candidate for the inverse of $[g]$ is $[g^{-1}]$ where $g^{-1}(x) := g(x)^{-1}$. Since this is just post-composing with the continuous inverse for U it is easy to check this is well-defined at the level of homotopy classes. Thus $[g^{-1}] + [g] = [g^{-1} g] = [1_X]$, so we have defined an inverse.

Let's see what is going on from the point of view of vector bundles. If $E$ is a vector bundle on $SX$, then as $SX$ is compact, we have that $E \subset SX \times \mathbb{C}^n$ for some $n$ sufficiently large. Let $E^\perp$ be the complement of $E$ and let $g^\perp$ be its clutching function. Then it can be checked that $g \cdot g^\perp = g^\perp \cdot g$ is a clutching function for the trivial bundle, whence $[g^\perp] = -[g]$. However, it is not immediately clear why constructing the inverse this way is well-defined. We will revisit this issue later when we construct $K^0(X)$. It turns out that there is a multiplication structure on $K_{-1}(X)$ making it a ring, but this is also slightly opaque to define without first describing $K_0$.

Note that while there is a bijective correspondence between complex N-bundles and $[X, U(N)]$, there is not quite such a correspondence at the level of $K^{-1}(X)$ as we have effectively "squished" all of the trivial bundles to a single class. What to do about this at the level of $K^0$ will be the distinction between reduced and unreduced K-theory.

## 13. Lecture 13

The computation of $K^{-1}(X)$ will lie out of reach for most basic examples without developing more machinery. However, for the moment, let's compute one extremely important example, $K^{-1}(S^1)$. First note that if X and Y are homotopy equivalent, then $K^{-1}(X) \cong K^{-1}(Y)$. Thus if $K^{-1}(X) = 0$ for X contractible. Since U is path connected, we also have that $K_{-1}(S^0) = 0$.

**Proposition 13.1** ("Baby Bott"). $K^{-1}(S^1) \cong [S^1, U(1)] \cong \mathbb{Z}$ *as groups.*

Any element of $K^{-1}(S^1)$ is represented by a map $g : S^1 \to U(n)$ for some $n$ sufficiently large. The map $p_n : U(n) \to S^{2n-1}$ which reads off the rightmost column vector is a principal $U(n-1)$-bundle, i.e., $U(n)/U(n-1) \cong S^{2n-1}$. We have that $p_n \circ g : S^1 \to S^{2n-1}$ is contractible to the constant map to the coset of the identity if $n \geq 2$. Since any fiber bundle has the **homotopy lifting property** (see Hatcher, Algebraic Topology, p. 379), this lifts to give a homotopy from $g$ to a map $g' : S^1 \to U(n-1)$. We see that the desired conclusion holds inductively.

Thinking about this a little more, it follows that $\pi_1(SU(n)) = 0$ for all $n$, whence explicitly $g$ is homotopic to $\det(g)$. As a side note it can be shown that $U(n)$ is **rationally homotopy equivalent** to $S^1 \times S^3 \times \cdots \times S^{2n-1}$.

The canonical generator of $K^{-1}(S^1)$ is $[z \mapsto z]$ which as a clutching function gives the canonical line bundle $\gamma^1$ over $\mathbb{C}P^1 \cong S^2$. This is known as the **Bott element**.

Note that the previous proposition can be easily restated as:

**Proposition 13.2.** *Every complex vector bundle over $S^2$ is a line bundle or a sum of a line bundle and a trivial bundle.*

### 13.1. Vector Bundles and Homotopy Theory.

We have already established that $\mathrm{Vect}^k_{\mathbb{C}}(SX) \cong [X, U(k)]$. Is there a target space such that $\mathrm{Vect}^k_{\mathbb{C}}(X) \cong [X, Y]$? Whether you appreciate being asked leading questions or not, the answer is, 'yes,' and we will now set about demonstrating this.

For natural numbers $k \leq n$, recall the complex Grassmannian manifold $\mathrm{Gr}_{k,n} = \mathrm{Gr}_{k,n}(\mathbb{C})$ is the space of all k-dimensional subspaces of $\mathbb{C}^n$. On can topologize this for instance by realizing $\mathrm{Gr}_{k,n}$ as the space $\mathcal{P}_k(n) = \{p \in M_n(\mathbb{C}) : p = p^* = p^2, \mathrm{tr}(p) = k\}$ of all rank k projections with the restriction of any of the natural topologies on $M_n(\mathbb{C})$. Remember there is a canonical (tautological) vector bundle $(E_{k,n}, \gamma_{k,n})$ defined as all pairs $(V, \vec{v})$ where $V \subset \mathbb{C}^n$ is a k-dimensional subspace and $\vec{v} \in V$.

The **Stiefel manifold** $V_{k,n}$ is defined to be the set of all orthonormal k-tuples of vectors in $\mathbb{C}$. The projection map $p : V_{k,n} \to \mathrm{Gr}_{k,n}$ which sends such a k-tuple to its span clearly has the structure of a principal $U(k)$-bundle. In fact, $V_{k,n} \otimes_{U(k)} \mathbb{C}^k$ is isomorphic to the tautological bundle on $Gr_{k,n}$.

There is a natural embedding $Gr_{k,n} \hookrightarrow Gr_{k,n+1}$. We define $Gr_{k,\infty}$ to be the inductive limit. Similarly, we construct the inductive limits of the Stiefel manifolds, $V_{k,\infty}$. The inductive limit of the projections $p_n : V_{k,n} \to Gr_{k,n}$ gives a projection $p : V_{k,\infty} \to Gr_{k,\infty}$ again forming a principal $U(k)$-bundle.

We write $\mathrm{Vect}^{k,n}_{\mathbb{C}}(X)$ to be the set of all complex $k$-bundles over $X$ which admit an $n$-frame. We also write $[X, Gr_{k,\infty}]_n$ to be all homotopy classes in $[X, Gr_{k,\infty}]$ represented by maps whose image lies in $Gr_{k,n}$.

**Proposition 13.3.** *Let $X$ be a compact space. There is a bijective correspondence between $[X, Gr_{k,\infty}]_n$ and $\mathrm{Vect}^{k,n}_{\mathbb{C}}(X)$ given by $[f : X \to Gr_{k,n}] \mapsto f^*(\gamma_{k,n})$.*

This is a well-defined map by the Bundle Homotopy Theorem, so we need only construct an inverse. For $E \in \mathrm{Vect}^{k,n}_{\mathbb{C}}(X)$ we can realize $E \subset \mathbb{C}^n$. Let $g_E : X \to Gr_{k,n}$ be the map $g_E : x \mapsto E_x \subset \mathbb{C}^n$ be the map which sends $x$ to its fiber in $\mathbb{C}^n$. It is readily apparent that $g_E^*(\gamma_{k,n})$ is isomorphic to $E$.

Conversely, for $f \in [X, Gr_{k,n}]$ and $E = f^*(\gamma_{k,n})$ a choice of $n$-frame for $E$ is implicit in the pull-back over the tautological bundle for which $g_E = f$. Thus, while the map $g_E$ depends on the embedding $E \hookrightarrow X \times \mathbb{C}^n$, we are fine as long as we can show the homotopy class of the map is well defined. To this end, let $h_1, h_2 : E \to X \times \mathbb{C}^n$ be two $n$-frames for the bundle $E$. For $t \in [0, 1]$ and $e \in E$, define $h_t : E \to \mathbb{C}^n \oplus \mathbb{C}^n \cong \mathbb{C}^{2n}$ to be $h_t(e) := (x, (1-t)\vec{v}_x \oplus t\vec{w}_x)$ where $(x, \vec{v}_x) = h_1(e)$ and $(x, \vec{w}_x) = h_2(e)$. It is easy to check that each $h_t$ is an $n$-frame. Thus $E \mapsto [g_E] \in [X, Gr_{k,2n}]$ is a well-defined map from $\mathrm{Vect}^{k,n}_{\mathbb{C}}(X)$ to $[X, Gr_{k,2n}]$.

From this it immediately follows that:

**Proposition 13.4.** *For any compact space $X$ there is a bijective correspondence between $\mathrm{Vect}^{k}_{\mathbb{C}}(X)$ and $[X, Gr_{k,\infty}]$.*

The gluing construction effectively provides a correspondence between isomorphism classes of complex $k$-bundles over $X$ and principal $U(k)$-bundles over $X$. Thus, the previous proposition can thus be restated as there is a bijective correspondence between isomorphism classes of principal $U_k$-bundles and homotopy classes $[X, Gr_{k,\infty}]$ obtained by pulling back the canonical principal $U(k)$-bundle $p : V_{k,\infty} \to Gr_{k,\infty}$.

## 14. Lecture 14

In general, for a topological group $G$, we say that a space $BG$ is a **classifying space** for $G$ if there is a principal $G$-bundle $p : EG \to BG$ such that for any compact space $X$ there is a bijective correspondence between isomorphism classes of principal $G$-bundles over $X$ and $[X, BG]$ obtained by pulling back the bundle $EG$. By previous proposition we can take $Gr_{k,\infty}$ as $BU(k)$. By work on Milnor, every topological group $G$ admits a classifying space. The construction is analogous to the familiar bar construction in homology theory, so is quite unwieldy to work with in practice.

For the infinite unitary group $U$, we obtain a model of $B\,U$ by noting there is a natural map $Gr_{k,n} \to Gr_{k+1,n+1}$ obtained by summing with a one-dimensional space an this is well behaved with respect to the inductive limits giving an inclusion $\mathrm{Gr}_{k,\infty} \hookrightarrow \mathrm{Gr}_{k+1,\infty}$. We define $BU$ to be the inductive limit. Again, by the gluing construction, every principal U-bundle over a compact space is represented by a U-valued cocycle on a finite cover which must reduce to a $U(N)$-cocycle for some $N$ sufficiently large again by compactness. In other words every principal U-bundle $P$ is of the form $P = P' \times_{U(N)} U$ for some principal $U(N)$-bundle $P'$ for some $N$ sufficiently large. It is apparent that the map $[X, \mathrm{Gr}_{k,\infty}] \to [X, \mathrm{Gr}_{k+1}, \infty]$ induced by the inclusion $\mathrm{Gr}_{k,\infty} \hookrightarrow \mathrm{Gr}_{k+1,\infty}$ identifies on the principal bundle side with the map $P \mapsto P \times_{U(k)} U(k+1)$. Since $\varinjlim [X, \mathrm{Gr}_{k,\infty}] \cong [X, \varinjlim \mathrm{Gr}_{k,\infty}] = [X, BU]$ for any compact space $X$, this shows that $BU$ is indeed a classifying space for U.

14.1. **The Grothendieck Construction.** Given $A$ an abelian monoid, we can define an abelian group $K(A)$ and a homomorphism $\iota_A : A \to K(A)$ such that for any homomorphism $\varphi : A \to B$ of abelian monoids, there is a homomorphism of abelian groups $K(\varphi) : K(A) \to K(B)$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & B \\ {\scriptstyle \iota_A}\downarrow & & {\scriptstyle \iota_B}\downarrow \\ K(A) & \xrightarrow{K(\varphi)} & K(B) \end{array}$$

The group $K(A)$ can be constructed as follows. Define an equivalence relation $\sim$ on $A \times A$ by $(a_1, b_1) \sim (a_2, b_2)$ if $a_1 + b_2 + c = a_2 + b_1 + c$ for some $c \in A$. This relation is trivially reflexive and is symmetric by commutativity of the sum. Further if $(a_1, b_1) \sim (a_2, b_2) \sim (a_3, b_3)$ then $a_1 + b_2 + c + b_3 + d = a_2 + b_1 + c + b_3 + d = a_3 + b_2 + b_1 + c + d$ for some $c, d \in A$, so setting $e = b_2 + c + d$ we have $(a_1, b_1) \sim (a_3, b_3)$.

The additive structure on $K(A)$ is $[a_1, b_1] + [a_2, b_2] = [a_1 + a_2, b_1 + b_2]$ which is straightforward to check is well-defined as associative. The zero class is $[a, a]$ for any $a \in A$ and the inverse class to $[a, b]$ is $[b, a]$. The map $\iota_A : A \to K(A)$ is given by $a \mapsto [a, 0]$ where $0$ is the identity in $A$. Finally, if $\varphi : A \to B$ is a homomorphism of abelian monoids, then $[a, b] \mapsto [\varphi(a), \varphi(b)]$ induces a group homomorphism $K(\varphi) : K(A) \to K(B)$ which can be easily seen to fit into the commutative diagram above. Sometimes we write the class $[a, b]$ as the formal difference $[a] - [b]$.

Moreover, suppose that $A$ is equipped with a second associative operation $*$ which distributes over summation. Then $K(A)$ is equipped with a ring structure by $([a]-[b]) * ([c] - [d]) = [a * c] - [a * d] - [b * c] + [b * d] = [a * c + b * d] - [a * d + b * c]$. The ring structure is also natural in the sense of fitting into the commutative diagram above.

**Exercise 14.1.** Show that the Grothendieck construction applied to the non-negative integers yields the integers.

14.2. **Defining** $K^0(X)$ **and** $\widetilde{K}^0(X)$**.** There are two natural functors from the category compact, hausdorff topological spaces to abelian groups defined as terms of $\mathrm{Vect}_{\mathbb{C}}(X)$. One is motivated is motivated directly from vector bundles and associated operations on them, while the other is motivated more from the homotopy-theoretic perspective as outline in the previous section. We will begin with the former.

For a compact, hausdorff topological space $X$, $\mathrm{Vect}_{\mathbb{C}}(X)$ is an abelian monoid under direct sum (the $0$-dimensional vector bundle is the identity). We thus define $K^0(X) := K(\mathrm{Vect}_{\mathbb{C}})$. More over the tensor product of vector bundles is seen to distribute over summation, so $K^0(X)$ is a ring.

The basic properties of the functor $K^0(\cdot)$ naturally follow from operations on vector bundles. If $X$ and $Y$ are homotopy equivalent, then $K^0(X) \cong K^0(Y)$ by the Bundle Homotopy Theorem. We have the $K^0(pt) \cong \mathbb{Z}$ as a ring, since all vector bundles are trivial, so $E \mapsto \dim(E)$ gives an isomorphism between $\mathrm{Vect}_{\mathbb{C}}(pt)$ and $\mathbb{N} \cup \{0\}$ which is compatible with summation and the tensor product. If $f : X \to Y$ is continuous, then for $E, F \in \mathrm{Vect}_{\mathbb{C}}(Y)$ we have that $f^*(E \oplus F) \cong f^*E \oplus f^*F$ and $f^*(E \otimes F) \cong f^*E \otimes f^*F$, whence the pullback induces a (unital) ring homomorphism $K^0(f) : [E] - [F] \mapsto [f^*E] - [f^*F]$ from $K^0(Y)$ to $K^0(X)$, i.e., the functor $K^0$ is contravariant.

If we specialize to the inclusion $pt \to X$, this induces a map $K^0(X) \to K^0(pt)$ which, in the case that $X$ is connected, is independent of the choice of map as every fiber belonging to a single connected component must have the same dimension. Let us denote this map $\epsilon : K^0(X) \to \mathbb{Z}$. Since $X$ has at least one vector bundle in every dimension we have that:

**Proposition 14.2.** *$\epsilon : K^0(X) \twoheadrightarrow \mathbb{Z}$ is a surjective ring homomorphism.*

Note that since $X$ is compact, every vector bundle admits a frame, whence for every bundle $E \hookrightarrow X \times \mathbb{C}^N$, there is the complementary bundle $E^\perp$ so that $E \oplus E^\perp \cong 1_N$, the trivial bundle of dimension $N$. Thus for any $[E] - [F] \in K^0(X)$ we have that $[E] - [F] = [E \oplus F^\perp] - [F \oplus F^\perp] = [E'] - [1_n]$. If $X$ is connected, the kernel of the homomorphism $\epsilon$ can thus be identified with all classes $[E] - [1_n]$ where $\dim(E) = n$. Our first definition of $\widetilde{K}^0(X)$ is the kernel of $\epsilon$ with the inherited ring structure.

**Proposition 14.3.** *$\epsilon : K^0(S^1) \cong \mathbb{Z}$ is a ring isomorphism, whence $\widetilde{K}^0(S^1) \cong 0$.*

From the cocycle perspective, we have that every complex $n$-bundle on $S^1$ is given by a clutching map $f \in [S^0, U(n)]$, since $U(n)$ is path connected, it follows that every complex bundle over $S^1$ is trivial, whence the isomorphism.

Of course, we could define the **real K-group** $K^0_{\mathbb{R}}(X)$ by applying the Grothendieck construction to $\mathrm{Vect}_{\mathbb{R}}(X)$ and $\widetilde{K}^0_{\mathbb{R}}(S^1)$ is defined similarly. This is again a ring in the analogous way. If we compute $K^0_{\mathbb{R}}(S^1)$, we can still do so by reducing to clutching functions, this time in $\varinjlim [S^0, O(n)]$. Since $O(n)$ has exactly two connected components for each $n$, we have that any clutching function $f : S^0 \to O(n)$ is homotopic to $\det(f) : S^0 \to \{-1, 1\}$. Whence every nontrivial real vector bundle over $S^1$ is isomorphic to the Möbius bundle. Thus $K^0_{\mathbb{R}}(S^1) \cong \mathbb{Z}[x]/(x^2 - 1)$ with the natural ring structure. This gives that $\widetilde{K}^0_{\mathbb{R}}(X) \cong \mathbb{Z}/2\mathbb{Z}$; however, the product structure is trivial (all products are zero). This will probably be the last time in these notes that we discuss real K-theory, so I hope you enjoyed this digression, but not too much.

We now turn to the second approach to defining the $\widetilde{K}^0$ functor via homotopy theory. We say two vector bundles $E, F \in \mathrm{Vect}_{\mathbb{C}}(X)$ are **stably equivalent** (written $E \sim_s F$) if

$E \oplus 1_n \cong F \oplus 1_m$ for some $m, n \in \mathbb{N} \cup \{0\}$. Note that the vector bundles need not have the same dimension. There is a natural monoid structure on the stable equivalences classes defined by $[E]_s + [F]_s := [E \oplus F]_s$. It is straightforward to check that this operation is well-defined, commutative, and associative. The class $[1]_s$ of the trivial line bundle is a obviously the unit. We now construct an additive inverse. Let $h_1 : E \to X \times \mathbb{C}^m$ and $h_2 : E \to \mathbb{C}^n$ be two frames of the same vector bundle. By extending we have that $h_1 \oplus 0_{m+2n}, 0_{2m+n} \oplus h_2 : E \to X \times \mathbb{C}^{2m+2n}$ are two orthogonal frames for $E$ in the same bundle, so by essentially the same reasoning as in the proof of Proposition 13.3, we have that $h_1(E)^\perp \oplus 1_n = (h_1 \oplus 0_n(E))^\perp \cong (0_m \oplus h_2(E))^\perp = 1_m \oplus h_2(E)^\perp$, whence $E \mapsto h_1(E)^\perp$ is well-defined at the level of stable equivalence classes. Thus for any splitting $E \oplus F \cong X \times \mathbb{C}^l$ we have that $[E]_s + [F]_s = [1_l]_s$, so $[F]_s = -[E]_s$ is a well-defined inverse, whence stable equivalence classes of vector bundles form an abelian group. Let's call the group of stable equivalence classes of complex vector bundles over $X$ by $G(X)$.

**Proposition 14.4.** *There is an isomorphism of groups* $\theta : \widetilde{K}^0(X) \to G(X)$ *given by* $\theta : [E] - [1_n] \mapsto [E]_s$.

The map $\mathrm{Vect}_{\mathbb{C}}(X) \ni E \mapsto [E]_s \in G(X)$ is clearly a surjective homomorphism of monoids, whence induces a surjective homomorphism $K^0(X) \twoheadrightarrow G(X)$. Moreover, the trivial bundles are all sent to the identity in $G(X)$ so this maps factors to a surjective homomorphism $\widetilde{K}^0(X) \twoheadrightarrow G(X)$. We need only show this map is also injective. If $E$ and $F$ are comlex bundles of respective dimensions $m$ and $n$, then we have that $[E]_s = [F]_s$ if and only if $E \oplus 1_n \oplus 1_k \cong F \oplus 1_m \oplus 1_k$ for some $k$, whence by definition of the Grothendieck construction we have that $[E] - [1_m] = [F] - [1_n]$.

**Exercise 14.5.** As we noted before, $\widetilde{K}^0(X)$ inherits a ring structure as an ideal in $K^0(X)$. Write down this ring structure in terms of $G(X)$.

We are now ready to give a characterization of $\widetilde{K}^0(X)$ in terms of homotopy theory.

**Proposition 14.6.** *There is a bijective correspondence between* $[X, BU]$ *and stable equivalence classes of complex vector bundles over* $X$.

By Proposition 13.4 there is a bijective correspondence between $\mathrm{Vect}^k_{\mathbb{C}}(X)$ and $[X, \mathrm{Gr}_{k,\infty}]$. By the proof of that result is straightforward that the map $E \mapsto E \oplus 1$ sending $\mathrm{Vect}^k_{\mathbb{C}}(X)$ into $\mathrm{Vect}^{k+1}_{\mathbb{C}}(X)$ corresponds with the induced map $[X, \mathrm{Gr}_{k,\infty}] \to [X, \mathrm{Gr}_{k+1,\infty}]$ given by the inclusion $\mathrm{Gr}_{k,\infty} \hookrightarrow \mathrm{Gr}_{k+1,\infty}$.

In particular $[X, BU]$ has the structure of an abelian group. We conclude this section by outlining how to construct the group structure internally. Note that we can place an abelian monoid structure on $[X, BU]$. First note there is a natural map $\sigma : BU \times BU \to BU$ by embedding the first copy into the odd coordinates and the second copy into the even coordinates. We define a sum for for $[f], [g] \in [X, BU]$ by $([f] + [g])(x) := [\mu(f(x), g(x))]$. Again, it is not too hard to demonstrate that this is operation is well-defined, commutative, and associative. Given a splitting $E \oplus E^\perp \cong X \times \mathbb{C}^n$, we have for the maps $f_E(x) := E_x \in \mathrm{Gr}_{k,n}$ and $f_{E^\perp}(x) := E_x^\perp \in \mathrm{Gr}_{n-k,n}$ that $[f_E] + [f_{E^\perp}] = [1_n]$ giving the inverse.

## 15. Lecture 15

15.1. **Morse Theory.** [*This lecture was given by Jun Choi*]

## 16. Lecture 16

16.1. **Exact sequences of abelian groups and splittings.** In this section all capital Roman letters will denote abelian groups. We say that sequence of homomorphisms $A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} A_n$ is **exact** if $\mathrm{im}(f_i) = \ker(f_{i+1})$ for all $i = 1, \ldots, n-1$. A homomorphism $A \xrightarrow{f} B$ is said to be **split** if there is a homomorphism $B \xrightarrow{g} A$ such that $f \circ g = \mathrm{id}_B$. The following proposition will be crucial:

**Proposition 16.1.** *Let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ be an exact sequence. Then $f$ and $g$ are both split if and only if $B \cong A \oplus C$. Moreover the same conclusion holds if and only if either $f$ or $g$ is split.*

One direction is trivial. Suppose that $s$ and $t$ split $f$ and $g$ respectively. For $b \in B$ write $b = (b - tg(b)) + tg(b)$ then $b - tg(b) \in \ker(g) = \mathrm{im}(f)$. Moreover, the fact that $g$ splits implies that $g$ restricted to $\mathrm{im}(t)$ is injective, whence $\mathrm{im}(f) \cap \mathrm{im}(t) = \{0\}$ by exactness. By exactness at $C$, $t$ is injective, and exactness at $A$ implies $f$ is injective. Thus we have that $B$ is generated by the images of two injective maps which have trivial intersection. Since $B$ is abelian this implies that $B \cong \mathrm{im}(f) \oplus \mathrm{im}(t) \cong A \oplus C$. A similar argument applies when considering $f$ and its splitting $s$.

**Exercise 16.2.** Complete the proof of this proposition.

As an immediate consequence it follows that:

**Proposition 16.3.** $K^0(X) \cong \widetilde{K}^0(X) \oplus \mathbb{Z}$.

We have that $0 \to \widetilde{K}^0(X) \to K^0(X) \xrightarrow{\epsilon} \mathbb{Z} \to 0$ is exact and the inclusion $K^0(pt) \to K^0(X)$ given by the map $X \to pt$ provides a splitting for $\epsilon$.

16.2. **Computing $K^0(S^2)$.** We will now describe $K^0(S^2)$ as a ring. First, somewhat confusingly we write $\widetilde{K}^{-1}(X) := K^{-1}(X)$. The reason for this will become apparent later. For the moment we just state:

**Proposition 16.4.** *For any compact, hausdorff space $X$, we have that $\widetilde{K}^0(SX)$ and $\widetilde{K}^{-1}(X)$ are isomorphic as abelian groups.*

As we noted earlier we have $[SX, BU(k)] \leftrightarrow \mathrm{Vect}^k_{\mathbb{C}}(SX) \leftrightarrow [X, U(k)]$, so by passing to direct limits we have that $\widetilde{K}^0(SX) \cong \widetilde{K}^{-1}(X)$ as sets. However, it is not too hard to see that the group operations we defined on $[SX, BU]$ and $[X, U]$ are compatible under this identification. Note we can use the ring structure on $\widetilde{K}^0$ to define a ring structure on $\widetilde{K}^{-1}$.

**Proposition 16.5.** *As abelian groups* $\widetilde{K}^0(S^2) \cong \widetilde{K}^{-1}(S^1) \cong \mathbb{Z}$ *and*

$$K^0(S^2) \cong \mathbb{Z} \oplus \mathbb{Z} = \{m[1] + n[\gamma] : m, n \in \mathbb{Z}\}$$

*where* $[1]$ *is the class of the trivial line bundle and* $[\gamma]$ *is the class of the tautological line bundle over* $\mathbb{C}P^1 \cong S^2$. *Moreover,* $[\beta] := [\gamma] - [1] \in \widetilde{K}^0(S^2)$ *corresponds with the Bott element (generator) of* $\widetilde{K}^{-1}(S^1)$.

This follows directly from the splitting $K^0(S^2) \cong \widetilde{K}^0(S^2) \oplus \mathbb{Z}$ discussed above.

We now describe the ring structure on $\widetilde{K}^0(S^2)$. It suffices to understand how $[\gamma]$ behaves under taking powers. For $m \in \mathbb{N}$, let $\gamma^m$ be the line bundle which is $m$-fold tensor power of the tautological line bundle $\gamma$. Since $\gamma$ is obtained by clutching over $S^1$ by the map $z \mapsto z$, we have that $\gamma^m$ is obtained by clutching over $S^1$ by the map $z \mapsto z^m$. We can extend this definition of $\gamma^m$ to all integers, e.g., $\gamma^{-1}$ corresponds to clutching over $z \mapsto z^{-1}$, etc. For $\gamma^m \oplus \gamma^n$ the clutching function is

$$z \mapsto \begin{pmatrix} z^m & 0 \\ 0 & z^n \end{pmatrix} \sim \begin{pmatrix} z^{m+n} & 0 \\ 0 & 1 \end{pmatrix}$$

whence we have the relation $[\gamma^m] + [\gamma^n] = [\gamma^{m+n}] + [1]$. Iterating this we see that

$$m[\gamma] = [\gamma^m] + (m-1)[1] = [\gamma]^m + (m-1)[1]$$

holds for all integers.

For a variable $x$, let $\mathbb{Z}[x]$ denote the ring of polynomials in $x$ with coefficients in $\mathbb{Z}$. If $p(x) \in \mathbb{Z}[x]$ we write $\mathbb{Z}[x]/p(x)$ to be the ring which is the quotient of $\mathbb{Z}[x]$ modulo the ideal generated by $p(x)$.

**Proposition 16.6.** *As a ring we have that* $K^0(S^2) \cong \mathbb{Z}[\gamma]/(\gamma - 1)^2 \cong \mathbb{Z}[\beta]/\beta^2$ *where the isomorphisms are given by* $[\gamma] \mapsto \gamma$ *and* $[\beta] \mapsto \beta$, *respectively.*

This follows easily from the computation

$$([\gamma] - [1])^2 = [\gamma]^2 - 2[\gamma] + 1 = (2[\gamma] - 1) - 2[\gamma] + 1 = 0.$$

16.3. **The Short Exact Sequence.** In this section we begin the work of extending $\widetilde{K}^0$ and $\widetilde{K}^{-1}$ to a generalized cohomology theory with higher (reduced) K-groups $\widetilde{K}^{-n}$ for all $n$. The main goal of this section is to establish:

**Proposition 16.7** (Short Exact Sequence). *If* $X$ *is a compact, hausdorff space and* $A \subset X$ *is closed then the sequence* $A \xrightarrow{i} X \xrightarrow{q} X/A$ *induces an exact sequence on reduced* $K$*-theory*

$$\widetilde{K}^0(X/A) \xrightarrow{q_*} \widetilde{K}^0(X) \xrightarrow{i_*} \widetilde{K}^0(A).$$

Clearly the pullback of any bundle over $X/A$ along $q$ is trivial when restricted to $A$. As $i_*$ is the map induced by restriction to $A$, this shows that $\mathrm{im}(q_*) \subset \ker(i_*)$. In the other direction, we will use the description of $\widetilde{K}^0$ as stable equivalence classes of vector bundles. If $E$ is a vector bundle over $X$ with $i_*[E]_s = [E|_A]_s = [1]_s$, then by replacing $E$ with $E \oplus 1_k$ for $k$ sufficiently large we can assume without loss of generality that $E|_A$ is a trivial bundle over $A$. In other words, we can assume that $E|_A = A \times \mathbb{C}^n$ for some $n$.

There is an obvious candidate for the preimage class of $E$ under $q_*$, namely the class of the quotient $E/\sim$ where $(x, \vec{v}) \sim (y, \vec{v})$ whenever $\vec{v} \in \mathbb{C}^n$ and $x, y \in A$. The only sublety is whether there is an open neighborhood of the point $A/A$ in $X/A$ for which $E/\sim$ trivializes. However, by Lemma 9.2 we can extend each of the $n$ pointwise linearly independent sections $s_1, \dots, s_n : A \to A \times \mathbb{C}^n$ on a common open neighborhood $U$ of $A$. Since these section vary continuously there is an open neighborhood $V$ of $A$ contained in $U$ where the sections remain pointwise linearly independent. These maps reduce to continuous maps $s_1', \dots, s_n' : V/A \to E/\sim|_{V/A}$ in the obvious way thus provide a local trivialization for $E/\sim$ at $A/A$. Thus $E/\sim$ is a vector bundle over $X/A$.

## 17. Lecture 17

17.1. **The Long Exact Sequence.** Given a compact, hausdorff, pointed space $(X, x_0)$ and a closed subspace $x_0 \in A \subset X$, we define the **relative (reduced) $K$-theory** to be $\widetilde{K}^0(X, A) := \widetilde{K}^0(X/A)$. Considering the reduced suspension $\Sigma X$ with the canonical inclusion $\Sigma A \subset \Sigma X$ we have that $\Sigma X/\Sigma A$ is homeomorphic to $\Sigma(X/A)$. (This is not quite true for the suspension.) Therefore we have that $\widetilde{K}^{-1}(X, A) := \widetilde{K}^0(\Sigma X, \Sigma A) \cong \widetilde{K}^0(\Sigma(X/A)) \cong \widetilde{K}^{-1}(X/A)$.

Analogusly, we can define the **higher (relative) $K$-groups** by $\widetilde{K}^{-n}(X) := \widetilde{K}^0(\Sigma^n X)$ and $\widetilde{K}^{-n}(X, A) := \widetilde{K}^0(\Sigma^n X, \Sigma^n A) \cong \widetilde{K}^0(\Sigma^n(X/A))$ for all $n \in \mathbb{N}$. Rephrasing the results from the previous lecture, this means that for each $n \in \mathbb{N}$ the sequence

$$\widetilde{K}^{-n}(X, A) \xrightarrow{q_*} \widetilde{K}^{-n}(X) \xrightarrow{i_*} \widetilde{K}^{-n}(A)$$

is exact. The main result of this section is that these sequences can be stitched together to form one infinite exact sequence of K-groups, the "long exact sequence" which should be familiar from general (co)homology theory.

**Proposition 17.1.** *There is a map $\partial : \widetilde{K}^{-1}(A) \to \widetilde{K}^0(X, A)$ so that the sequence*

$$\widetilde{K}^{-1}(X) \to \widetilde{K}^{-1}(A) \xrightarrow{\partial} \widetilde{K}^0(X, A) \to \widetilde{K}^0(X)$$

*is exact.*

The trick to building this map is the very useful one of turning a quotient into an inclusion via homotopy theory. For a (based) inclusion $Y \hookrightarrow X$ of compact, hausdorff, based spaces we have that $X/Y$ is homotopy equivalent to $X \cup CY$ where we identify the base of the cone with the image of $Y$, yielding a sequence of inclusions

$$Y \hookrightarrow X \hookrightarrow X \cup CY.$$

In this way we have reimagined the sequence $A \to X \to X/A$ in a way that we can now extend indefinitely by applying the operation of "coning off the previous thing" in the sequence! Thus we have

$$Y \hookrightarrow X \hookrightarrow X \cup CY \hookrightarrow (X \cup CY) \cup CX \hookrightarrow ((X \cup CY) \cup CX) \cup (X \cup CY) \hookrightarrow \cdots$$

and so on. The $(k+2)$-nd term in the sequence is thus homotopic the the quotient of the $(k+1)$-st term of the sequence by the $k$-th term, so the chain of groups we get by applying the $\widetilde{K}^0$ functor to the sequence is exact at each term.

We have that $(X \cup CY) \cup CX$ is homotopy equivalent to $\Sigma Y$ and $((X \cup CY) \cup CX) \cup (X \cup CY)$ is homotopy equivalent to $\Sigma X$ (note that $SX \cong CX/X$).

**Exercise 17.2.** Draw some pictures to verify this.

Applying these identifications we have an exact sequence

$$\cdots \to \widetilde{K}^0(\Sigma X) \to \widetilde{K}^0(\Sigma A) \to \widetilde{K}^0(X/A) \to \widetilde{K}^0(X) \to \widetilde{K}^0(A).$$

The map $\partial : \widetilde{K}^{-1}(A) \to \widetilde{K}^0(X, A)$ is idenitfied with the middle map in this sequence.

### 17.2. Applications of the long exact sequence.

## 18. Lecture 18

**18.1. Towards Bott Periodicity.** Let $X$ be a compact, hausdorff space. From previous discussion we know that $[X, \Omega\, BU] \cong [\Sigma X, BU] \cong [X, U]$. The first isomorphism is just the fact that the loop and reduced suspension functors are adjoint and the second isomorphism follows from the fact that both homotopy classes describe isomorphism classes of vector bundles on $\Sigma X$. As a consequence, note that $\pi_{k+1}(BU) \cong \pi_k(\Omega\, BU) \cong \pi_k(U)$ for all $k \in \mathbb{N}$. In fact the connection between these two spaces is much deeper. The goal of this section is to prove the following theorem:

**Proposition 18.1.** *For each $k$, the spaces $U(k)$ and $\Omega\, BU(k)$ are homotopy equivalent.*

Since inductive limits commute with the loop functor, this shows that $U$ and $\Omega\, BU$ are also homotopy equivalent.

Before sketching the proof, we discuss a bit of notation and some generalities. We will say that two topological spaces $X$ and $Y$ are **weakly homotopy equivalent** if there is a map $f : X \to Y$ which induces a bijection $f_* : \pi_0(X) \to \pi_0(Y)$ and isomorphisms $f_* : \pi_k(X) \to \pi_k(Y)$ for all $k \in \mathbb{N}$. It turns out that if both $X$ and $Y$ have the homotopic to CW complexes, then weak homotopy equivalence implies homotopy equivalence, so we will not have much reason to trouble ourselves with the distinction for the purpose of these notes.

For any topological group $G$, it can be shown there is a classifying space and a principal $G$-bundle $EG \to BG$ with $EG$ contractible. It can also be shown through abstract, homotopy-theoretic considerations that this implies that $G$ is always weakly homotopy equivalent to $\Omega BG$. In fact, as we will see shortly, the map $f : G \to \Omega BG$ can be written easily. We will use the stucture of $BU(k)$, along with a little linear algebra, to construct an explicit homotopy inverse to $f$.

Let $(B, b_0)$ be a based space. The **path space** $PB$ is the space of all continuous maps $f : [0, 1] \to B$ with $f(0) = b_0$. The path space is naturally a based space with base point the constant path at $b_0$. Let $F \to E \xrightarrow{p} B$ be a fibration of based spaces. This means that $p : (E, e_0) \to (B, b_0)$ is an $F$-fibration with $e_0 \in p^{-1}(b_0) \cong F$. For any based space $B$, the map $ev : PB \to B$ given by $ev : f \mapsto f(1)$ is a natural based fibration with $F = \Omega B$. Given such a based fibration with $E$ contractible, our first objective is to create a commutative diagram of based maps:

$$\begin{tikzcd} F \arrow[r] \arrow[d] & E \arrow[r, "p"] \arrow[d] & B \arrow[d, equal] \\ \Omega B \arrow[r] & PB \arrow[r, "ev"] & B \end{tikzcd}$$

To do so, since $E$ is contractible, let $h : E \times [0,1] \to E$ be a homotopy from the constant map to $e_0$ to the identity map on $E$. Then $x \mapsto h(x,t)$ gives a map $E \to PE$. The map $p : E \to B$ induces a map $p_* : PE \to PB$ which we can compose with the previous map to get the middle downward arrow in the diagram. Restricting this map to $p^{-1}(b_0) \cong F$ then gives the leftmost downward arrow. Applying this to the classifying space $B = BG$, this is the map $f : G \to \Omega BG$.

We will directly check that ths applies to the classifying bundle $V_{k,\infty} \to Gr_{k,\infty} = BU(k)$ which boils down to showing:

**Proposition 18.2.** *For each* $k$, $V_{k,\infty}$ *is contractible.*

Let $\mathbb{C}^\infty$ be the inductive limit of $(\mathbb{C}^n)$ under the standard inclusions, and let $\mathbb{C}^{ev}$ be all vectors supported on even coordinates. For each $n$ there is a path of unitaries which takes the copy of $\mathbb{C}^n$ in $\mathbb{C}^{2n}$ situated on the first $n$ coordinates to the copy situated on the even numbered coordinates, preserving the order. Applying inductive limits this yeilds a homotopy of $\mathbb{C}^\infty$ onto $\mathbb{C}^{ev}$ through orthogonality preserving maps. For each $k$, this induces a homotopy from $V_{k,\infty}$ to the submanifold $V_{k,ev}$ of all orthonormal $k$-tuples in $\mathbb{C}^{ev}$ by applying the homotopy to each element of a $k$-tuple. Now given an orthonormal $k$-tuple $v_1, \ldots, v_k \in \mathbb{C}^{ev}$ we have that $t \mapsto \sqrt{1-t}v_i + te_{2i-1}$ gives a homotopy through orthonormal $k$-tuples from $\{v_1, \ldots, v_k\}$ to $\{e_1, e_3, \ldots, e_{2k-1}\}$ where $e_i$ is the $i$th element of the canonical orthonormal basis for $\mathbb{C}^\infty$. We can then map this homotopy back into $V_{k,ev}$ showing that $V_{k,ev} \cong V_{k,\infty}$ is contractible.

In the case that $B = BU(k)$ we now describe how we can reverse the vertical arrows in the commutative diagram, constructing a homotopy inverse for the map $U(k) \to \Omega\, BU(k)$.

**Proposition 18.3.** *There is a map* $\upsilon : PB \to PE$ *of based spaces so that* $x \mapsto \upsilon(p)(1)$ *and its restriction to* $\Omega B$ *make the following diagram commute:*

$$\begin{tikzcd} F \arrow[r] & E \arrow[r, "p"] & B \\ \Omega B \arrow[r] \arrow[u] & PB \arrow[r, "ev"] \arrow[u] & B \arrow[u, equal] \end{tikzcd}$$

An element $p \in PB$ is a path $t \mapsto V_t$ of $k$-dimensional subspaces of $\mathbb{C}^N$ for some $N$ sufficiently large. The basepoint $e_0 \in E$ is a distinguished (ordered) orthonormal basis $(v_1, \ldots, v_k)$ for $V_0$. Let $pr_{s,t}$ be the orthogonal projection from $E_s$ onto $E_t$ for $s,t \in [0,1]$. For every $\delta > 0$, there is $\epsilon > 0$ sufficiently small such that whenever $|s-t| < \epsilon$ the image of any ordered orthonormal basis of $V_s$ under $\mathrm{pr}_{s,t}$ is $\delta$-close to an ordered orthonormal basis of $V_t$ and we can select such one canonically via the Gram-Schmidt process. Thus starting from an ordered orthonormal basis of $V_s$ we can find a continuous path of orthonormal bases for $V_t$ with $t < s + \epsilon$.

This not only shows that we can lift a path $p \in PB$ to a path $\tilde{p} \in PE$ but we can do so in a continuous way over $PB$. (To be sure we are sweeping a lot of small details under the rug here.) The map $p \mapsto \tilde{p}(1) \in E$ is the middle arrrow in our diagram. Restricting to $\Omega B$, we have that $\tilde{p}(1)$ is another ordered orthonormal basis $(w_1, \ldots, w_k)$ for $V_0$, but these are in one-to-one correspondence with elements of $U(k)$ by sending the basis to the unitary defined by $v_i \mapsto w_i$ for all $i = 1, \ldots, k$.

One can compare the two processes in order to convince oneself that they really do describe the inverse operations to each other, at least up to homotopy. (Sweep, sweep!)

## 19. Lecture 19

### 19.1. **Bott Periodicity: Bott's Proof.**

## 20. Lecture 20

### 20.1. **Bott Periodicity: Proof of Atiyah–Bott.**

## 21. Lecture 21

## 22. Lecture 22

## References

[1] M.F. Atiyah, *K-Theory*, W.A. Benjamin
[2] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*, Graduate Texts in Mathematics **82**, Springer.
[3] A. Hatcher, *Algebraic Topology*, Cambridge.
[4] A. Hatcher, *Vector Bundles and K-Theory*, manuscript.
[5] J.W. Milnor, *Morse Theory*, Princeton UP.
[6] J.W. Milnor and J.D. Stasheff, *Characteristic Classes*, Princeton UP.